OPTIMAL APPROXIMATION IN BANACH SPACES

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Introduction

Approximation theory is a central field of functional analysis that investigates the efficiency of representing complex functions through simpler, finite-dimensional subspaces. This graduation project explores the principles of optimal approximation within Banach spaces, with a primary focus on the concept of m-widths. Since their introduction by Kolmogorov, m-widths have become essential tools for evaluating and categorizing various numerical approximation methods.

The first part of this work lays out the foundational definitions of vector spaces, normed linear spaces, and the property of completeness that defines Banach spaces. We further examine topological properties, such as compactness and local compactness, which are necessary for ensuring the existence and uniqueness of best approximation elements. Central in our analysis is the Borsuk-Ulam theorem, which provides a powerful topological framework for deriving lower estimates of widths.

The second part of the project shifts toward applications in periodic function spaces. We define generalized smoothness classes, specifically the Weyl-Nagy classes, using Weyl derivatives and the modulus of continuity. By analyzing multiplier operators and convolution kernels, we aim to provide upper and lower bounds for approximation errors in L_p spaces. This theoretical development ultimately allows us to determine the optimality of specific approximation subspaces for these function classes.

1 General Definitions

Definition 1.1. *[1, p.50]* ***(Vector space).*** *A vector space (or linear space) over a field K is a nonempty set X whose elements $x, y, \ldots$ are called vectors, equipped with two algebraic operations: vector addition and multiplication of vectors by scalars, where scalars are elements of K.*

Vector addition assigns to each ordered pair (x, y) a vector $x + y$, the sum of x and y, satisfying commutativity and associativity, i.e., for all vectors:

$$x + y = y + x, \qquad x + (y + z) = (x + y) + z;$$

there exists a zero vector 0 and, for every vector x, an additive inverse $-x$, such that:

$$x + 0 = x, \qquad x + (-x) = 0.$$

Scalar multiplication assigns to each vector x and scalar α a vector αx, satisfying for all vectors x, y and scalars α, β:

$$\alpha(\beta x) = (\alpha\beta)x, \qquad 1x = x,$$

$$\alpha(x + y) = \alpha x + \alpha y, \qquad (\alpha + \beta)x = \alpha x + \beta x.$$

The field K is called the scalar field of X. When $K = \mathbb{R}$, the space X is called a real vector space; when $K = \mathbb{C}$, it is called a complex vector space.

Definition 1.2. *[1, p.58]* ***(Normed space, Banach space).*** *A normed space X is a vector space on which a norm is defined. A Banach space is a normed space that is complete with respect to the metric induced by its norm (see* (1.1) *below). A norm on a (real or complex) vector space X is a real-valued function whose value at $x \in X$ is written*

$$\|x\| \qquad \text{(read “norm of } x\text{”)}$$

and satisfies the following properties for all vectors x, $y \in X$ and every scalar α:

(N1) $\|x\| \geq 0$
(N2) $\|x\| = 0 \iff x = 0$
(N3) $\|\alpha x\| = |\alpha| \, \|x\|$

(N4) $\|x+y\| \leq \|x\|+\|y\|$ *(Triangle inequality).*

Every norm on X induces a metric d on X defined by

$$d(x,y)=\|x-y\| \qquad (x,y\in X), \tag{1.1}$$

called the metric induced by the norm. The resulting normed space is denoted by $(X,\|\cdot\|)$, or simply X.

Definition 1.3. *[1, p.77]* ***(Compactness).*** *A metric space X is called compact if every sequence in X contains a convergent subsequence. A subset $M \subset X$ is called compact if it is compact as a subspace of X, meaning every sequence in M has a subsequence converging to a point in M.*

Definition 1.4. *[1, p.82]* ***(Locally compact).*** *A metric space X is called locally compact if every point of X has a neighborhood that is compact.*

2 Widths and Best Approximation

2.1 Kolmogorov and Bernstein Widths

The concept of m-widths was introduced by Kolmogorov [2] in 1936 as a means of measuring the efficiency of numerical approximation methods. Let X be a Banach space with norm $\|\cdot\|$. For a symmetric set $A \subset X$, the Kolmogorov m-width $d_m(A,X)$ is defined by:

$$d_m(A,X)=\inf_{L_m\subset X}\sup_{x\in A}\inf_{y\in L_m}\|x-y\|,$$

where the outer infimum is taken over all m-dimensional subspaces L_m of X. The computation of m-widths is generally carried out in two stages: first, estimating the quantity

$$E(L_m,A,X)=\sup_{x\in A}\inf_{y\in L_m}\|x-y\|,$$

for a fixed subspace L_m to obtain a suitable upper bound, and second, establishing a lower bound for $d_m(A,X)$. The difficulty in deriving lower estimates comes from the requirement to consider every possible m-dimensional subspace

$L_m \subset X$. In 1960, Tikhomirov [7] proved a theorem on the diameter of a ball (see Theorem 2.7) by introducing a new topological method based on the Borsuk–Ulam theorem, which provided a systematic approach to such lower bounds. A simple proof of Theorem 2.7 is given below. We first recall some definitions. Let B denote the unit ball of a Banach space X, and let A be a compact, centrally symmetric subset of X. For an $(m+1)$-dimensional subspace $L_{m+1} \subset X$, the *Bernstein m-width* is defined by

$$b_m(A, X) = \sup\{L_{m+1} \subset X \mid \sup\{\epsilon > 0 \mid \epsilon B \cap L_{m+1} \subset A\}\}.$$

2.2 Cowidths

Cowidths arise in the study of optimal recovery problems. Given a metric (Banach) space (X, ϑ) and a coding set Y, let $A \subset X$ and let Θ denote a family of mappings $\theta : A \longrightarrow Y$. The associated *cowidth* is defined by:

$$co^{\Theta}(A, X) = \inf_{\theta \in \Theta} \sup_{y \in \theta(A)} \operatorname{diam}\{\theta^{-1}(y) \cap A\},$$

where

$$\theta^{-1}(y) = \{x \in X, \theta(x) = \theta(y)\},$$

and

$$\operatorname{diam}(B) = \sup\{\|x - y\| \mid x, y \in B\}.$$

In the particular case $Y = \mathbb{R}^m$ and $\Theta : A \longrightarrow \mathbb{R}^m$ a linear mapping, $\Theta = \mathcal{L}(A, \mathbb{R}^m)$, the resulting quantity is the *linear cowidth* $\lambda^m(A, X)$. A straightforward verification gives $\lambda^m = 2d^m$, where d^m denotes the Gelfand m-width:

$$d^m(A, X) = \inf\{L_{-m} \subset X \mid \sup\{\|x\| \mid x \in A \cap L_{-m}\}\},$$

with the infimum taken over all subspaces $L_{-m} \subset X$ of codimension m.

2.3 Functional and Operator of Best Approximation

We now turn to several classical results on the functionals and operators of best approximation. Let X be a Banach space with norm $\|\cdot\|_X = \|\cdot\|$. The lemmas that follow show that, under mild assumptions on M, the best approximation operator P is well-defined, unique, and sufficiently regular to play the role of the continuous odd map required in the application of the Borsuk–Ulam theorem. The *deviation* of a point $x \in X$ from a non-empty set $M \subset X$ is given by:

$$\begin{aligned} E(x) &= E(x, M) = E(x, M, X) \\ &= \inf\{\|x - y\| \mid y \in M\} \end{aligned} \tag{2.1}$$

which is called the *best approximation* of x from M. For a fixed $M \subset X$, the mapping $E : X \longrightarrow \mathbb{R}_+$ given by (2.1) is called the *best approximation functional*.

Lemma 2.1. *For any linear manifold $M \subset X$, the functional $E(\cdot, M)$ is uniformly continuous and subadditive:*

$$E(x_1 + x_2) \leq E(x_1) + E(x_2), \quad \forall x_1, x_2 \in X,$$

positively homogeneous:

$$E(ax) = |a|E(x), \quad \forall a \in \mathbb{R}$$

and is convex:

$$E(ax_1 + (1 - a)x_2) \leq aE(x_1) + (1 - a)E(x_2),$$

$$\forall a \in [0, 1], \forall x_1, x_2 \in X.$$

Proof. Fix $x_1, x_2 \in X$. For any $y \in M$ we have:

$$E(x_1) \leq \|x_1 - y\| = \|x_1 - x_2 + x_2 - y\| \leq \|x_1 - x_2\| + \|x_2 - y\|.$$

Taking the infimum over $y \in M$ gives:

$$E(x_1) \leq \|x_1 - x_2\| + E(x_2),$$

and hence

$$E(x_1) - E(x_2) \leq \|x_1 - x_2\|.$$

This proves the uniform continuity of E. For subadditivity, observe that for any $y_1, y_2 \in M$:

$$E(x_1 + x_2) \leq \|x_1 + x_2 - y_1 - y_2\| \leq \|x_1 - y_1\| + \|x_2 - y_2\|.$$

Minimizing the right-hand side over y_1 and y_2 gives $E(x_1 + x_2) \leq E(x_1) + E(x_2)$, which proves subadditivity. For $x \in X$ and $a \in \mathbb{R} \setminus \{0\}$ we have:

$$E(ax) = \inf_{y \in M} \|ax - y\| = |a| \inf_{y \in M} \left\| x - \frac{y}{a} \right\| = |a| E(x).$$

This establishes positive homogeneity, and since E is both subadditive and positively homogeneous, it is convex. $\square$

If there exists $y_0 \in M$ such that $E(x) = \|x - y_0\|$, then y_0 is called an *element of best approximation* for x in M. The set $M \subset X$ is called an *existence set* if every $x \in X$ admits at least one element of best approximation in M.

Lemma 2.2. *Any closed, locally compact subset $M \subset X$ is an existence set. In particular, every finite-dimensional subspace of X is an existence set.*

Proof. Let $x \in X \setminus M$ with $E(x) = c > 0$ (the case $x \in M$ is trivial). By the definition of the infimum, for each $n \in \mathbb{N}$ there exists $y_n \in M$ such that:

$$\|x - y_n\| < E(x) + \frac{1}{n}.$$

The sequence $\{y_n\}$ is bounded, since:

$$\|y_n\| = \|x - x + y_n\| \leq \|x\| + E(x) + \frac{1}{n} = \|x\| + c + \frac{1}{n}.$$

By the local compactness of M, there exists a subsequence $\{y_{n_m}\}$ converging to some y_0. Since M is closed, $y_0 \in M$. Clearly:

$$E(x) \leq \|x - y_{n_m}\| < E(x) + \frac{1}{n_m}, \quad m \in \mathbb{N}.$$

Letting $m \to \infty$ gives $\|x - y_0\| = E(x)$, so y_0 is an element of best approximation. $\square$

A norm on X is called *strictly convex* if for any $x, y \in X$ with $\|x\| = \|y\| = 1$, the inequality $\|ax + (1-a)y\| < 1$ holds for every $a \in (0, 1)$. Equivalently, the unit sphere of X contains no line segments.

Lemma 2.3. *Let M be a convex subset of a strictly convex normed space X. Then, for any $x \in X$, the element of best approximation in M, when it exists, is unique.*

Proof. Suppose there exist two distinct elements $y_1, y_2 \in M$ $(y_1 \neq y_2)$ that both yield a best approximation for $x \in X$:

$$E(x) = \|x - y_1\| = \|x - y_2\|.$$

By the convexity of M, the point $y_a = ay_1 + (1-a)y_2$ lies in M for any $a \in [0, 1]$, and:

$$\begin{aligned} E(x) \leq \|x - y_a\| &= \|a(x - y_1) + (1 - a)(x - y_2)\| \\ &\leq a\|x - y_1\| + (1 - a)\|x - y_2\| \\ &= aE(x) + (1 - a)E(x) = E(x). \end{aligned}$$

This would mean that the sphere $\{z \in X; \|x - z\| = E(x)\}$ contains the segment $y_a = ay_1 + (1 - a)y_2$ for $a \in [0, 1]$, contradicting strict convexity. □

A set $M \subset X$ is called a *Chebyshev set* if for every $x \in X$ there exists a unique element of best approximation. For such a set M, the best approximation operator (or metric projection) $P(x)$ is defined by:

$$E(x, M) = \|x - P(x)\|, \quad P(x) \in M.$$

Lemma 2.4. *If M is a locally compact Chebyshev set in X, then the operator P is continuous. Moreover, if M is a Chebyshev subspace, then P is homogeneous and, in particular, odd, so that $P(-x) = -P(x)$.*

Proof. Fix $x_0 \in X$ and let $x_m \to x_0$. Then:

$$\|P(x_m) - x_0\| \leq \|P(x_m) - x_m\| + \|x_m - x_0\| = E(x_m, M) + \|x_m - x_0\|.$$

By Lemma 2.1, the sequence $\{E(x_m, M)\}$ converges, so $\{P(x_m)\}$ is bounded. Suppose, for contradiction, that $P(x_m) \not\to P(x_0)$. Then by the local compactness of M there exists a subsequence $P(x_{m_n})$ with $\lim_{n\to\infty} P(x_{m_n}) = z \neq P(x_0)$. Since M is a Chebyshev set and therefore closed, $z \in M$. Passing to the limit as $n \to \infty$ in

$$\|x_{m_n} - P(x_{m_n})\| = E(x_{m_n}, M) \leq \|x_{m_n} - P(x_0)\|$$

gives $\|x_0 - z\| \leq \|x_0 - P(x_0)\|$, which means z is an element of best approximation for x_0 in M. This contradicts the assumption that M is a Chebyshev set, so

$P(x_m) \to P(x_0)$.

When M is a Chebyshev subspace, for any $a \in \mathbb{R}$ we have:

$$\|ax - P(ax)\| = |a|E(x, M) = \|ax - aP(x)\|,$$

which gives $P(ax) = aP(x)$. □

Take $M = M_m$ to be an m-dimensional Chebyshev subspace of the normed space X with basis $\{x_1, \ldots, x_m\}$. Then the best approximation operator can be written as:

$$P(x) = \sum_{k=1}^{m} \alpha_k(x) x_k. \tag{2.2}$$

From Lemma 2.4 we obtain:

Lemma 2.5. *The functionals $\alpha_k(x) : X \to M_m$ for $1 \leq k \leq m$ are homogeneous and continuous.*

Proof. By Lemma 2.4, $P(ax) = aP(x)$, which gives:

$$\sum_{k=1}^{m} \alpha_k(ax) x_k = \sum_{k=1}^{m} a\alpha_k(x) x_k.$$

The representation (2.2) is unique, so for every $a \in \mathbb{R}$ and $x \in X$ we have $\alpha_k(ax) = a\alpha_k(x)$, $1 \leq k \leq m$. Finally, since convergence in a finite-dimensional space M_m ($\dim M_m = m$) is equivalent to componentwise convergence and the operator $P : X \to M_m$ is continuous, the functionals α_k, $1 \leq k \leq m$, are continuous. □

2.4 Borsuk–Ulam Theorem and its Applications

The following theorem is a key result used frequently in the derivation of lower estimates for m-widths [8].

Theorem 2.6 (Borsuk-Ulam)**.** *Let X and Y be finite-dimensional Banach spaces over $\mathbb{R}$ or $\mathbb{C}$, with $\dim Y < \dim X$, and let $S = S(X) = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. Then for any continuous mapping $f : S \to Y$, there exists at least one element $x \in S$ such that $f(-x) = f(x)$. In particular, if f is an odd mapping, then there exists $x \in S$ such that $f(x) = 0$.*

First conjectured by Ulam and later proved by Borsuk, Theorem 2.6 can also be stated in the following equivalent form. Let Ω be an open, bounded, and symmetric neighborhood of $\mathbf{0}$ in $\mathbb{R}^m$, and let $F : \partial\Omega \to \mathbb{R}^{m-1}$ be a continuous odd mapping from the boundary $\partial\Omega$ to $\mathbb{R}^{m-1}$. Then there exists a point $x^* \in \partial\Omega$ such that $F(x^*) = 0$.

Theorem 2.7. *Let X_{n+1} be an arbitrary $(n+1)$-dimensional subspace of a real normed linear space X, and let $B(X_{n+1})$ denote the unit ball of X_{n+1}. Then,*

$$d_k(B(X_{n+1}), X) = 1, \quad 1 \leq k \leq m.$$

Proof. Clearly

$$\begin{aligned} d_m(B(X_{n+1}), X) &\leq d_{m-1}(B(X_{n+1}), X) \\ &\leq \cdots \leq d_0(B(X_{n+1}), X) = 1, \end{aligned}$$

so it remains to prove $d_m(B(X_{n+1}), X) \geq 1$. We will show that for any m-dimensional subspace $L_m \subset X$, there exists an element $x \in \partial B(X_{n+1})$ whose best approximation from L_m is zero. Let $\{x_1, \ldots, x_{m+1}\}$ and $\{z_1, \ldots, z_m\}$ be bases of X_{m+1} and L_m, respectively. Then every $x \in X_{m+1}$ and $z \in L_m$ admits the representation

$$x = \sum_{s=1}^{m+1} a_s x_s, \quad z = \sum_{s=1}^{m} b_s z_s.$$

It suffices to take $X = \operatorname{lin}\{X_{m+1}, L_m\}$, with $\dim X = l \leq 2m+1$. Choosing $\{y_1, \ldots, y_l\}$ as a basis of $X = \operatorname{lin}\{X_{m+1}, L_m\}$, every $x \in X$ can be written as

$$x = \sum_{s=1}^{l} c_s y_s.$$

If the norm on X is not strictly convex, we replace it with the norm

$$\|x\|_\epsilon = \|x\| + \epsilon \left(\sum_{s=1}^{l} |c_s|^2 \right)^{\frac{1}{2}} \tag{2.3}$$

which is strictly convex. Since $\dim X < 2m+1$, letting $\epsilon \to 0$ preserves the conclusion of the theorem. Hence we may assume that the norm on X is strictly convex, in which case the best approximation operator is unique, continuous, and odd. The domain

$$\Omega = \left\{ (a_1, \ldots, a_{m+1}) : x = \sum_{s=1}^{m+1} a_s x_s, \|x\| < 1 \right\}$$

is an open, bounded, and symmetric neighborhood of $\mathbf{0}$ in $\mathbb{R}^{m+1}$. For any $a \in \partial\Omega$, let $F(a) = (b_1, \ldots, b_m) \in \mathbb{R}^m$ be the vector of coefficients which corresponds to the best approximation of

$$x = \sum_{s=1}^{m+1} a_s x_s \in \partial B(X_{m+1})$$

from the subspace L_m. By Lemma 2.5, $F : \partial\Omega \to \mathbb{R}^m$ is a continuous and odd mapping from $\partial\Omega$ to $\mathbb{R}^m$. The Borsuk–Ulam theorem then guarantees the existence of some

$$x^* = \sum_{s=1}^{m+1} a_s^* x_s, \quad \|x^*\| = 1,$$

whose best approximation from L_m is the zero element. □

Combining Theorem 2.7 with the notion of Bernstein m-widths yields

Corollary 2.8. *Let A be a compact and symmetric set in a Banach space X. Then*

$$d_m(A, X) \geq b_m(A, X), \quad m = 0, 1, \ldots.$$

3 Function Classes

The general theory developed in Section 2 applies to arbitrary Banach spaces. In what follows, we specialize to the case $X = L_p$, the space of 2π-periodic p-integrable functions on $(-\pi, \pi)$. We introduce the relevant function classes through generalized derivatives and the modulus of continuity, and study their approximation properties in terms of the n-widths defined above.

3.1 The Space L_0 and Basic Function Sets

To analyze the approximation properties of periodic functions, we first define the standard spaces of 2π-periodic functions.

Definition 3.1. *[3, p.101] (L_p space) The space L_p, $1 \le p < \infty$, is the space of 2π-periodic functions $f(t)$ on $(-\pi, \pi)$, with norm*

$$\|f\|_{L_p} = \left(\int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{1/p}.$$

Definition 3.2. *Let $L(-\pi, \pi)$ denote the space of 2π-periodic functions that are summable on the interval $[-\pi, \pi]$. The subspace L_0 is defined as the set of functions in $L(-\pi, \pi)$ whose integral over the period is zero:*

$$L_0 = \left\{ f \in L(-\pi, \pi) : \int_{-\pi}^{\pi} f(t) dt = 0 \right\}. \tag{3.1}$$

Functions in L_0 are characterized by Fourier series of the form

$$S[f] = \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

3.2 Weyl Derivatives and $\mathcal{J}_r^r$ Classes

Differentiation can be extended beyond integer orders using the Weyl integral representation. This extension enables a more precise classification of functions according to how well they can be approximated.

Definition 3.3. *(Weyl Derivative) For a fixed $r > 0$, suppose the Fourier series of a function $f \in L(-\pi, \pi)$ can be represented via an auxiliary function $\varphi \in L_0$ as follows:*

$$S[f] = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \frac{1}{\pi k^r} \int_{-\pi}^{\pi} \varphi(x - t) \cos\left(kt - \frac{r\pi}{2}\right) dt. \tag{3.2}$$

In this case, φ is called the Weyl derivative of f of order r, denoted by $\varphi = D^r f = f^{(r)}$. The collection of all such functions f constitutes the class $\mathcal{J}_r^r$.

3.3 Conjugate Functions and $\mathcal{J}_{r+1}^r$ Classes

The construction of Weyl–Nagy classes $\mathcal{J}_\beta^r$ necessitates the introduction of trigonometric conjugate functions, which correspond to a phase shift in the Fourier coefficients.

Definition 3.4. *(Conjugate Function) Let $f \in L(-\pi, \pi)$ have the Fourier series $S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$. The conjugate series is defined as*

$\tilde{S}[f] = \sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx)$. *If this series is the Fourier series of some function* $\tilde{f} \in L(-\pi, \pi)$*, then* $\tilde{f}$ *is the conjugate function of* f*, often represented by the singular integral:*

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cot\frac{t}{2}dt. \tag{3.3}$$

Definition 3.5. *The set* $\mathcal{J}_{r+1}^{r}$ *is defined as the collection of functions whose Fourier series has the form:*

$$S[\tilde{f}] = \sum_{k=1}^{\infty}\frac{1}{\pi k^r}\int_{-\pi}^{\pi}\varphi(x-t)\cos\left(kt - \frac{r+1}{2}\pi\right)dt, \quad \varphi \in L_0. \tag{3.4}$$

Note: These foundational definitions provide the necessary framework to understand the general Weyl–Nagy classes $\mathcal{J}_{\beta}^{r}$ *as linear combinations of the* $\mathcal{J}_{r}^{r}$ *and* $\mathcal{J}_{r+1}^{r}$ *classes.*

3.4 Integration of Fourier Series

The following theorem is used in the derivation of the convolution representation of functions in L_{β}^{ψ}.

Theorem 3.6. *[3, p.114] Every Fourier series* $S[f]$ *can be integrated within any limits, regardless of whether it converges or not. This implies that the sum of the series of integrals of terms of a Fourier series is always equal to the integral of a function* $f(\cdot)$*.*

3.5 Modulus of Continuity and the Class H_ω

To quantify the smoothness of functions in the studied spaces, we use the modulus of continuity, which is an essential tool for determining the rate of convergence of approximation processes.

Definition 3.7. *(Modulus of Continuity) Let* X *be a Banach space of* 2π*-periodic functions (such as* C *or* L_p*). For any* $f \in X$ *and* $\delta > 0$*, the modulus of continuity is defined as:*

$$\omega(f, \delta)_X = \sup_{|h| \le \delta} \|f(\cdot + h) - f(\cdot)\|_X. \tag{3.5}$$

The modulus of continuity characterizes the maximum oscillation of the function within a distance δ.

Definition 3.8. *(The Class H_ω) Let $\omega(\delta)$ be a given majorant (a continuous, non-decreasing, subadditive function such that $\omega(0) = 0$). The functional class H_ω is defined as the set of all functions $f \in X$ satisfying the condition:*

$$\omega(f, \delta)_X \leq \omega(\delta), \quad \forall \delta > 0. \tag{3.6}$$

When $\omega(\delta) = \delta^\alpha$ for some $0 < \alpha \leq 1$, this class coincides with the classical Lipschitz (or Hölder) class, denoted by $\mathrm{Lip}\,\alpha$.

3.6 Weyl–Nagy Classes

The exposition in this section follows Stepanets [3, Ch. 3, §6].

The results of Subsections 3.2 and 3.3 motivate the introduction of the sets $\mathcal{J}^r_\beta$, consisting of those $f(\cdot)$ for which, with $r > 0$ and $\beta \in (-\infty, \infty)$ fixed,

$$S[f] = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \frac{1}{\pi k^r} \int_{-\pi}^{\pi} \varphi(x-t) \cos\Big(kt - \frac{\beta\pi}{2}\Big)\, dt \tag{3.7}$$

$$\begin{aligned} S[f] = \frac{a_0(f)}{2} &+ \cos\frac{(\beta - r)\pi}{2} \sum_{k=1}^{\infty} \frac{1}{\pi k^r} \int_{-\pi}^{\pi} \varphi(x-t) \cos\Big(kt - \frac{r\pi}{2}\Big)\, dt \\ &+ \sin\frac{(\beta - r)\pi}{2} \sum_{k=1}^{\infty} \frac{1}{\pi k^r} \int_{-\pi}^{\pi} \varphi(x-t) \cos\Big(kt - \frac{r+1}{2}\pi\Big)\, dt, \end{aligned} \tag{3.8}$$

with $\varphi \in L_0$. To this representation we associate an operator $D^r_\beta : \mathcal{J}^r_\beta \to L_0$ via (3.7).

For $\beta = r + 2k$ with $k \in \mathbb{Z}$, one recovers $\mathcal{J}^r_\beta = \mathcal{J}^r_r$ from Subsection 3.2. For $\beta = r + 1 + 2k$, the classes $\mathcal{J}^r_\beta$ and $\mathcal{J}^r_{r+1}$ of Subsection 3.3 agree up to constant terms. For the remaining values of β, the elements of $\mathcal{J}^r_\beta$ are precisely linear combinations of conjugate pairs taken from $\mathcal{J}^r_r$ and $\mathcal{J}^r_{r+1}$.

These classes were first introduced and studied by Nagy. Accordingly, the function $\varphi(\cdot)$ in (3.7) is referred to as the Weyl–Nagy (r, β)-derivative of $f(\cdot)$, written $f^r_\beta(\cdot) = D^r_\beta f$. A direct calculation shows that

$$S[f^r_\beta] = \sum_{k=1}^{\infty} k^r \left(a_k(f) \cos\!\left(kx + \frac{\beta\pi}{2}\right) + b_k(f) \sin\!\left(kx + \frac{\beta\pi}{2}\right)\right). \tag{3.9}$$

In fact, (3.9) provides an alternative description of $\mathcal{J}^r_\beta$: it consists exactly of those $f(\cdot)$ for which the series on the right is itself the Fourier series of some integrable function. This equivalence follows by inserting (3.9) into (3.7) and carrying out elementary manipulations. We therefore have, for every $r > 0$ and $\beta \in (-\infty, \infty)$,

$$\mathcal{J}^r_\beta = \left\{ f(\cdot) : \; D^r_\beta f = f^r_\beta(\cdot) \in L_0 \right\}. \tag{3.10}$$

When the series in (3.7) converges, its sum is taken as the function $f(\cdot)$ regarded as an element of $\mathcal{J}^r_\beta$.

The classes of Weyl–Nagy differentiable functions are defined as follows. If $f \in \mathcal{J}^r_\beta$ and $f^r_\beta \in \mathfrak{N}$, we say that $f(\cdot)$ belongs to the class $\mathcal{J}^r_\beta \mathfrak{N}$. For a specific class $\mathfrak{N}$, we write $\mathcal{J}^r_\beta \mathfrak{N} = W^r_\beta \mathfrak{N}$.

Consequently, $W^r_\beta \mathfrak{N} = W^r_r \mathfrak{N}$ for $\beta = r + 2k$ with $k \in \mathbb{Z}$, while $W^r_\beta \mathfrak{N} = W^r_{r+1} \mathfrak{N} = \widetilde{W}^r_r \mathfrak{N}$ for $\beta = r + 1 + 2k$.

3.7 Classes $L^\psi_\beta \mathfrak{N}$

The exposition in this section follows Stepanets [3, Ch. 3, §7].

Replacing the factor k^{-r} in (3.7) and (3.9) by an arbitrary sequence $\psi(k)$ indexed by a natural variable produces a new family of periodic-function classes. For suitable choices of ψ, these classes recover $W^r_\beta \mathfrak{N}$ (and hence $W^r_r \mathfrak{N}$ and $W^r \mathfrak{N}$), while for other choices they extend to functions that lie outside the scope of the classical scale. Note that $W^r_r \mathfrak{N} = W^r \mathfrak{N}$ for $r \in \mathbb{N}$, and $W^r_\beta \mathfrak{N} = W^r_r \mathfrak{N}$ when $\beta = r$.

Let $f \in L(-\pi, \pi)$ have Fourier series $S[f]$, fix an arbitrary function $\psi(k)$ of a natural variable, and let $\beta \in (-\infty, \infty)$. Assume that the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left(a_k(f) \cos\!\left(kx + \frac{\beta\pi}{2}\right) + b_k(f) \sin\!\left(kx + \frac{\beta\pi}{2}\right)\right) \tag{3.11}$$

is itself the Fourier series of some element of $L(-\pi, \pi)$. The function so obtained is referred to as the (ψ, β)-derivative of $f(\cdot)$ and is denoted $f^\psi_\beta(\cdot)$; we write L^ψ_β for the collection of all f for which such a derivative exists.

For $f \in L^{\psi}_{\beta}$, an application of Theorem 3.6 followed by elementary manipulations yields

$$\frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \frac{\psi(k)}{\pi} \int_{-\pi}^{\pi} f^{\psi}_{\beta}(x-t) \cos\Big(kt - \frac{\beta\pi}{2}\Big)\, dt = S[f]. \tag{3.12}$$

The converse also holds: whenever the Fourier series of some $f(\cdot)$ takes the form

$$S[f] = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \frac{\psi(k)}{\pi} \int_{-\pi}^{\pi} \varphi(x-t) \cos\Big(kt - \frac{\beta\pi}{2}\Big)\, dt, \tag{3.13}$$

with $\varphi \in L_0$, then f belongs to L^{ψ}_{β}. The verification amounts to checking that the Fourier series of $\varphi(\cdot)$ matches (3.11). Indeed, if (3.13) holds, then for each $k = 1, 2, \ldots$,

$$\begin{aligned} a_k(f) &= \psi(k)\left(a_k(\varphi)\cos\frac{\beta\pi}{2} - b_k(\varphi)\sin\frac{\beta\pi}{2}\right), \\ b_k(f) &= \psi(k)\left(a_k(\varphi)\sin\frac{\beta\pi}{2} + b_k(\varphi)\cos\frac{\beta\pi}{2}\right). \end{aligned} \tag{3.14}$$

Solving for $a_k(\varphi)$ and $b_k(\varphi)$,

$$\begin{aligned} a_k(\varphi) &= \frac{1}{\psi(k)}\left(a_k(f)\cos\frac{\beta\pi}{2} + b_k(f)\sin\frac{\beta\pi}{2}\right), \\ b_k(\varphi) &= \frac{1}{\psi(k)}\left(b_k(f)\cos\frac{\beta\pi}{2} - a_k(f)\sin\frac{\beta\pi}{2}\right). \end{aligned} \tag{3.15}$$

A comparison of (3.15) and (3.11) confirms that $\varphi(\cdot)$ has a Fourier series of the required form.

It follows that L^{ψ}_{β} may be identified with the set $\tilde{L}^{\psi}_{\beta}$ of functions in $L(-\pi, \pi)$ whose Fourier series take the form (3.13) for some $\varphi \in L_0$. In this situation, $\varphi(\cdot)$ is again called the (ψ, β)-derivative of f, and the operator $L^{\psi}_{\beta} \to L_0$ defined by (3.13) is written D^{ψ}_{β}, so that $D^{\psi}_{\beta} f = f^{\psi}_{\beta}$.

Taking $\psi(k) = k^{-r}$ with $r > 0$ recovers $L^{\psi}_{\beta} = \mathcal{J}^{r}_{\beta}$. When the series in (3.12) converges, we identify its sum with $f(\cdot)$ as an element of L^{ψ}_{β}.

For $\mathfrak{N} \subset L(-\pi, \pi)$, we say that f belongs to $L^{\psi}_{\beta}\mathfrak{N}$ if $f \in L^{\psi}_{\beta}$ and $f^{\psi}_{\beta} \in \mathfrak{N}$. Each triple $(\psi, \beta, \mathfrak{N})$ thus determines a class $L^{\psi}_{\beta}\mathfrak{N}$, with the special case $\psi(k) = k^{-r}$ producing $L^{\psi}_{\beta}\mathfrak{N} = W^{r}_{\beta}\mathfrak{N}$.

We say that $f(\cdot)$ is the convolution of two elements $h(\cdot)$ and $g(\cdot)$ of $L(-\pi,\pi)$ if it admits the representation

$$f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x-t)\,g(t)\,dt,$$

written briefly as $f = h * g$, where $g(t)$ plays the role of the convolution kernel.

We first check that $f(x)$ is finite for almost every x. Without loss of generality, assume that $h(\cdot)$ and $g(\cdot)$ are nonnegative — the general case follows by decomposing each function into its positive and negative parts. The product $h(x-t)g(t)$ is then measurable as a function of (x,t), and being nonnegative, its iterated integrals may be computed in either order. We therefore obtain

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx &= \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h(x-t)\,g(t)\,dt\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\left(\int_{-\pi}^{\pi} h(x-t)\,dx\right)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\,dt\int_{-\pi}^{\pi} h(t)\,dt.\end{aligned} \tag{3.16}$$

Hence $f \in L(-\pi,\pi)$; in particular, f is finite throughout $(-\pi,\pi)$ and clearly periodic. To compute its Fourier coefficients, suppose

$$S[h] = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \tag{3.17}$$

$$S[g] = \frac{a_0'}{2} + \sum_{k=1}^{\infty}(a_k' \cos kx + b_k' \sin kx). \tag{3.18}$$

According to (3.16), $h(x-t)g(t)$ is summable over $-\pi \le x,\ t \le \pi$, so the order of integration may be exchanged, yielding

$$\begin{aligned}a_k(f) &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h(x-t)\,g(t)\cos kx\,dt\,dx \\ &= \frac{1}{\pi^2}\int_{-\pi}^{\pi} g(t)\int_{-\pi}^{\pi} h(x)\cos k(x+t)\,dx\,dt = a_k a_k' - b_k \cdot b_k', \quad k = 0,1,\dots\end{aligned}$$

An analogous computation gives $b_k(f) = a_k' b_k + a_k b_k'$ for $k = 1,2,\dots$ Combining these formulas yields the following theorem:

Theorem 3.9. *[3, p.123] Let $h, g \in L(-\pi,\pi)$. Then the convolution $f = h * g$ also belongs to $L(-\pi,\pi)$, and provided that* (3.17) *and* (3.18) *hold,*

$$S[f] = \frac{a_0 a_0'}{2} + \sum_{k=1}^{\infty}\big((a_k a_k' - b_k b_k')\cos kx + (a_k' b_k + a_k b_k')\sin kx\big). \tag{3.19}$$

The classes $L^{\psi}_{\beta}\mathfrak{N}$ are naturally studied through the action of multiplier operators of the form Λ associated with the sequence $\psi(k)$. To estimate n-widths of such classes, one needs sharp control over the operator norm of Λ acting on L_p spaces. The following lemma provides such an estimate and serves as a key tool in the width computations to follow.

Lemma 3.10. *[4][5] Let $1 < p < \infty$, then*

$$\left\|\Lambda\middle|L_p \to L_p\right\| \le \chi_p \left(\sum_{k=1}^{\infty} |\lambda(k) - \lambda(k+1)| + \sup_{m\in\mathbb{N}} |\lambda(m)|\right),$$

where

$$\chi_p = 1 + 2\begin{cases}\cot\dfrac{\pi}{2p}, & 2 < p < \infty,\\[2mm] \tan\dfrac{\pi}{2p}, & 1 < p \le 2.\end{cases}$$

Proof. Let $U : \varphi \to \tilde{\varphi}$, where

$$U\varphi := \tilde{\varphi} \sim \sum_{k=1}^{\infty} -b_k(\varphi)\cos kx + a_k(\varphi)\sin kx$$

and

$$S_m^*(\varphi, x) = \frac{1}{2}(S_m(\varphi,x) + S_{m-1}(\varphi,x)).$$

Then

$$S_m^*(\varphi,x) = \sin mx\,\tilde{g}_m(x) - \cos mx\,\tilde{h}_m(x) = \sin mx\,Ug_m(x) - \cos mx\,Uh_m(x),$$

where

$$g_m(x) := f(x)\cos mx, \quad h_m(x) := f(x)\sin mx.$$

Clearly,

$$\left\|S_m^*\middle|L_p \to L_p\right\| \le 2\left\|U\middle|L_p \to L_p\right\|,$$

and

$$\left\|S_m - S_m^*\right|L_p \to L_p\right\| \le 1.$$

Hence, for any $m \in \mathbb{N}$ we get

$$\begin{aligned}\left\|S_m\right|L_p \to L_p\right\| &= \left\|S_m - S_m^* + S_m^*\right|L_p \to L_p\right\| \\ &\le 1 + \left\|S_m^*\right|L_p \to L_p\right\| \le 1 + 2\left\|U\right|L_p \to L_p\right\|. \end{aligned} \tag{3.20}$$

It is known [6] that

$$C_p = \left\|U\right|L_p \to L_p\right\| = \begin{cases} \cot\dfrac{\pi}{2p}, & 2 < p < \infty, \\ \tan\dfrac{\pi}{2p}, & 1 < p \le 2. \end{cases} \tag{3.21}$$

Comparing (3.20) and (3.21) we get

$$\sup_{m\in\mathbb{N}}\left\|S_m\right|L_p \to L_p\right\| \le 1 + 2\begin{cases} \cot\dfrac{\pi}{2p}, & 2 < p < \infty, \\ \tan\dfrac{\pi}{2p}, & 1 < p \le 2. \end{cases} := \chi_p. \tag{3.22}$$

Application of Abel transform to $S_m\Lambda\varphi$ yields

$$\begin{aligned} S_m\Lambda\varphi(x) &= \sum_{k=1}^{m}\lambda(k)\big(a_k(\varphi)\cos kx + b_k(\varphi)\sin kx\big) \\ &= \sum_{k=1}^{m-1}(\lambda(k) - \lambda(k+1))S_k(\varphi, x) + \lambda(m)S_m\varphi(x). \end{aligned} \tag{3.23}$$

Comparing (3.22) and (3.23) we find

$$\begin{aligned} \sup_{m\in\mathbb{N}}\left\|S_m\Lambda\right|L_p \to L_p\right\| &\le \chi_p \sup_{m\in\mathbb{N}}\left(\sum_{k=1}^{m-1}|\lambda(k) - \lambda(k+1)| + |\lambda(m)|\right) \\ &\le \chi_p\left(\sum_{k=1}^{\infty}|\lambda(k) - \lambda(k+1)| + \sup_{m\in\mathbb{N}}|\lambda(m)|\right). \end{aligned} \tag{3.24}$$

Consequently, by the Banach–Steinhaus Theorem we get the proof. □

4 Applications and Main Results

Having established the function classes $L^{\psi}_{\beta}\mathfrak{N}$ and the multiplier estimate of Lemma 3.10, we are now in a position to derive sharp estimates for n-widths of these classes. The central result of this section is Theorem 4.3, which provides a two-sided asymptotic estimate for the Bernstein and Kolmogorov widths of W^{ψ}_{β} in L_p.

Proposition 4.1. *Let $1 < p < \infty$, $\beta \in \mathbb{R}$, and let $\psi(k)$, $k \in \mathbb{N}$, be an arbitrary function of a natural variable. Let Λ denote the multiplier operator associated with ψ, and let $U : f \mapsto \tilde{f}$ denote the trigonometric conjugate operator. Define the unit ball*

$$U_p = \{\varphi \in L_p : \|\varphi\|_p \leq 1\}.$$

Then every function $f \in L^{\psi}_{\beta}$ with (ψ, β)-derivative $\varphi \in U_p$ admits the representation $f = \Phi\varphi$, where

$$\Phi = \cos\frac{\beta\pi}{2} \cdot \Lambda + \sin\frac{\beta\pi}{2} \cdot \Lambda U. \tag{4.1}$$

Proof. By definition of the class L^{ψ}_{β}, the Fourier series of f has the form (see (3.13))

$$S[f] = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \frac{\psi(k)}{\pi} \int_{-\pi}^{\pi} \varphi(x-t) \cos\Big(kt - \frac{\beta\pi}{2}\Big)\, dt,$$

where $\varphi \in L_0$ is the (ψ, β)-derivative of f. We expand the cosine using the angle subtraction formula:

$$\cos\Big(kt - \frac{\beta\pi}{2}\Big) = \cos kt \cdot \cos\frac{\beta\pi}{2} + \sin kt \cdot \sin\frac{\beta\pi}{2}.$$

Substituting into (3.13) and separating the two terms gives

$$\begin{aligned} S[f] = \frac{a_0(f)}{2} &+ \cos\frac{\beta\pi}{2} \sum_{k=1}^{\infty} \frac{\psi(k)}{\pi} \int_{-\pi}^{\pi} \varphi(x-t) \cos kt\, dt \\ &+ \sin\frac{\beta\pi}{2} \sum_{k=1}^{\infty} \frac{\psi(k)}{\pi} \int_{-\pi}^{\pi} \varphi(x-t) \sin kt\, dt. \end{aligned} \tag{4.2}$$

We now identify each sum. Recall that the multiplier operator Λ acts as

$$\Lambda\varphi \sim \sum_{k=1}^{\infty} \psi(k)\big(a_k(\varphi) \cos kx + b_k(\varphi) \sin kx\big),$$

so that

$$\frac{\psi(k)}{\pi}\int_{-\pi}^{\pi}\varphi(x-t)\cos kt\,dt$$

is precisely the k-th term of $\Lambda\varphi$. Hence the first sum in (4.2) equals $\Lambda\varphi(x)$.

For the second sum, recall that the trigonometric conjugate operator $U : f \mapsto \tilde{f}$ satisfies

$$U\varphi \sim \sum_{k=1}^{\infty}\bigl(-b_k(\varphi)\cos kx + a_k(\varphi)\sin kx\bigr),$$

so that

$$\frac{\psi(k)}{\pi}\int_{-\pi}^{\pi}\varphi(x-t)\sin kt\,dt$$

is the k-th term of $\Lambda U\varphi(x)$. Hence the second sum in (4.2) equals $\Lambda U\varphi(x)$.

Substituting back into (4.2) yields

$$f(x) = \cos\frac{\beta\pi}{2}\cdot\Lambda\varphi(x) + \sin\frac{\beta\pi}{2}\cdot\Lambda U\varphi(x),$$

which is precisely (4.1).

It remains to verify that this representation is well-defined for $\varphi \in U_p$. By the Riesz theorem [9] (see (3.21)), the conjugate operator U is bounded on L_p for $1 < p < \infty$. Therefore $U\varphi \in L_p$ whenever $\varphi \in L_p$, and both $\Lambda\varphi$ and $\Lambda U\varphi$ are well-defined. □

Theorem 4.2. *Let $\psi(k)$ be a decreasing sequence. Then*

$$\sup_{f\in L_{\psi,\beta}U_p}\|f - S_n(f)\|_p \le C_p\,\psi(n+1).$$

Proof. Let I denote the identity operator on L_p. Using representation (4.1) we write

$$f = \cos\frac{\beta\pi}{2}\cdot\Lambda\varphi + \sin\frac{\beta\pi}{2}\cdot\Lambda U\varphi$$

Then:

$$\begin{aligned}\sup_{f\in L_{\psi,\beta}U_p}\|f - S_n(f)\|_p &= \left\|\left(\cos\tfrac{\beta\pi}{2}\cdot\Lambda + \sin\tfrac{\beta\pi}{2}\cdot\Lambda U\right)\varphi - S_n\left(\cos\tfrac{\beta\pi}{2}\cdot\Lambda + \sin\tfrac{\beta\pi}{2}\cdot\Lambda U\right)\varphi\right\|_p \\ &= \left\|\left(\cos\tfrac{\beta\pi}{2}\cdot\Lambda + \sin\tfrac{\beta\pi}{2}\cdot\Lambda U\right)(I - S_n)\varphi\right\|_p \\ &\le \|\Lambda_1(I - S_n)|L_p \to L_p\| + \|\Lambda_2(I - S_n)|L_p \to L_p\|\cdot\|U|L_p \to L_p\|,\end{aligned}$$

where $\Lambda_1 = \cos\frac{\beta\pi}{2}\cdot\Lambda$ and $\Lambda_2 = \sin\frac{\beta\pi}{2}\cdot\Lambda$.

The multiplier sequences of $\Lambda_1(I - S_n)$ and $\Lambda_2(I - S_n)$ are computed as follows:

$$\Lambda_1(I - S_n) = \cos\frac{\beta\pi}{2}\begin{cases}\psi(k+1), & k \geq n,\\ 0, & 1 \leq k \leq n,\end{cases} \tag{4.3}$$

$$\Lambda_2(I - S_n) = \sin\frac{\beta\pi}{2}\begin{cases}\psi(k+1), & k \geq n,\\ 0, & 1 \leq k \leq n.\end{cases} \tag{4.4}$$

Applying Lemma 3.10 we get:

$$\|\Lambda_1(I - S_n) \mid L_p \to L_p\| \leq C_p^{(1)}\,\psi(n+1), \tag{4.5}$$

$$\|\Lambda_2(I - S_n) \mid L_p \to L_p\| \leq C_p^{(2)}\,\psi(n+1). \tag{4.6}$$

Hence,

$$\sup_{f \in L_{\psi,\beta}U_p} \|f - S_n(f)\|_p \leq C_p\,\psi(n+1).$$

□

Theorem 4.3. *Let $\psi(k)$ be a decreasing sequence of real numbers and $\psi(k) \neq 0$, $k \in \mathbb{N}$. Then*

$$C_p^{(1)}\,\psi(n) \leq b_{2n}(L^{\psi}_{\beta}U_p, L_p) \leq d_{2n-1}(L^{\psi}_{\beta}U_p, L_p) \leq C_p^{(2)}\,\psi(n)$$

for any $n \in \mathbb{N}$ and $1 < p < \infty$.

Proof. The upper bounds follow from Theorem 4.2. Let Φ^{-1} be the inverse of Φ on the space $\mathcal{T}_{2n+1} = \text{lin}\{1, \cos kx, \sin kx,\ 1 \leq k \leq n\}$. Clearly,

$$\Phi^{-1} = \cos\frac{\beta\pi}{2}\,\Lambda^{-1}_{n+1} - \sin\frac{\beta\pi}{2}\,\Lambda^{-1}_{n+1}U,$$

where

$$\Lambda^{-1}_{n+1} = \begin{cases}(\psi(k))^{-1}, & 1 \leq k \leq n+1,\\ 0, & k > n+1,\end{cases}$$

Clearly

$$\begin{aligned}
\Phi^{-1}\Phi &= \Big(\cos\frac{\beta\pi}{2}\,\Lambda_{n+1}^{-1} - \sin\frac{\beta\pi}{2}\,\Lambda_{n+1}^{-1}U\Big)\Big(\cos\frac{\beta\pi}{2}\,\Lambda + \sin\frac{\beta\pi}{2}\,\Lambda U\Big) \\
&= \cos^2\frac{\beta\pi}{2}\,\Lambda_{n+1}^{-1}\Lambda + \cos\frac{\beta\pi}{2}\sin\frac{\beta\pi}{2}\,\Lambda_{n+1}^{-1}\Lambda U \\
&\quad - \sin\frac{\beta\pi}{2}\cos\frac{\beta\pi}{2}\,\Lambda_{n+1}^{-1}U\Lambda - \sin^2\frac{\beta\pi}{2}\,\Lambda_{n+1}^{-1}U\Lambda U \\
&= \cos^2\frac{\beta\pi}{2}\,I + \cos\frac{\beta\pi}{2}\sin\frac{\beta\pi}{2}\,U - \sin\frac{\beta\pi}{2}\cos\frac{\beta\pi}{2}\,U - \sin^2\frac{\beta\pi}{2}\,U^2 \\
&= \cos^2\frac{\beta\pi}{2}\,I - \sin^2\frac{\beta\pi}{2}\,(-I) \\
&= \Big(\cos^2\frac{\beta\pi}{2} + \sin^2\frac{\beta\pi}{2}\Big)I = I,
\end{aligned}$$

where we used $U^2 = -I$.

Direct calculation shows

$$\begin{aligned}
&\|\Phi^{-1} \mid L_p \cap \mathcal{T}_{2n+1} \to L_p \cap \mathcal{T}_{2n+1}\| \\
&\quad \le \|\Lambda_{n+1}^{-1} \mid L_p \cap \mathcal{T}_{2n+1} \to L_p \cap \mathcal{T}_{2n+1}\|\big(1 + \|U \mid L_p \to L_p\|\big).
\end{aligned}$$

Since $\psi(k)$ is decreasing, by Lemma 3.10:

$$\|\Lambda_{n+1}^{-1} \mid L_p \cap \mathcal{T}_{2n+1} \to L_p \cap \mathcal{T}_{2n+1}\| \le 2\chi_p\,\psi(n+1).$$

Consequently, $U_p \cap \mathcal{T}_{2n+1} \subset \Lambda U_p$, and therefore

$$\|\Phi^{-1} \mid L_p \cap \mathcal{T}_{2n+1} \to L_p \cap \mathcal{T}_{2n+1}\| \le 2\chi_p\,\psi(n+1)(1 + C_p).$$

Hence,

$$\frac{\psi(n+1)}{2\chi_p(1+C_p)}\,U_p \cap \mathcal{T}_{2n+1} \subset L_\beta^\psi U_p.$$

By Corollary 2.8:

$$b_{2n}(L_\beta^\psi U_p,\, L_p) \ge \frac{1}{2\chi_p(1+C_p)}\,\psi(n+1). \qquad \square$$

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