AN INVESTIGATION ON DUAL NUMBERS: ALGEBRAIC PROPERTIES, MATRIX REPRESENTATIONS, AND APPLICATIONS

Math 490 - Graduation Project

2025-2026 Spring Semester

Author
Ata GÜR

Supervisor
Dr. Neslihan Ayşen ÖZBAY

Department of Mathematics

Çankaya University

Abstract

Dual numbers, expressed in the form $a+b\varepsilon$ where $\varepsilon^2 = 0$, provide an algebraic framework for various mathematical and computational problems. This project explores the fundamental algebraic properties of dual numbers and investigates their structural characteristics. We examine the ring properties of dual numbers along with their ideal structures. Then, we investigate their representation through 2×2 real matrices and some basic properties of this matrix ring. Furthermore, the relationship between dual numbers and Taylor expansions is analyzed, and some applications of this expansion on dual numbers are presented. Our study also investigates dual functions and their derivatives. The study concludes with a brief overview of dual numbers in automatic differentiation.

The purpose of this study is to investigate the algebraic structure and fundamental properties of dual numbers. In this project, dual numbers are examined from the perspectives of ring theory, vector spaces, matrix representations, and Taylor expansions.

Today, dual numbers are used in many areas of mathematics, engineering, and computer science.

One of their most important applications is automatic differentiation, where derivatives of functions can be computed efficiently and accurately. For this reason, dual numbers are widely used in optimization algorithms and machine learning.

Dual numbers are also used in robotics and kinematics to represent rotations and translations simultaneously. In computer graphics and geometric modeling, they help describe motions and transformations in three-dimensional space.

In addition, dual numbers appear in physics, mechanics, sensitivity analysis, and numerical computations involving first-order approximations and infinitesimal changes.

Dual numbers are algebraic objects of the form

$$a + b\varepsilon \quad \text{where } \varepsilon^2 = 0,\ \varepsilon \neq 0.$$

They form a commutative algebra that shares similarities with complex numbers, yet exhibits fundamentally different behavior due to the nilpotent nature of ε.

Unlike complex numbers, where $i^2 = -1$, dual numbers satisfy a degeneracy condition $\varepsilon^2 = 0$, which leads to the existence of zero divisors and prevents the structure from being a field.

Dual numbers appear naturally in: automatic differentiation, kinematics (screw theory) and geometry

Historically, they were introduced by William Kingdon Clifford in the 19th century as part of a broader study of geometric algebras.

Contents

Preliminaries

In this section we give some basic definitions

Definition 0.1 (Ring)**.** *A set R with two operations $+$ and $*$ such that:*

- *$(R, +)$ is an abelian group*
- *multiplication is associative*
- *distributive laws hold*

Definition 0.2 (Subring)**.** *Let R be a ring. A subset $S \subseteq R$ is called a subring if:*

- *S is closed under addition and multiplication,*
- *S contains the additive identity,*
- *S contains additive inverses.*

Definition 0.3 (Ring Isomorphism)**.** *Let R and S be rings. A map*

$$\varphi : R \to S$$

is called a ring isomorphism if:

$$\varphi(a + b) = \varphi(a) + \varphi(b)$$

$$\varphi(ab) = \varphi(a)\varphi(b)$$

and φ is bijective.
If such a map exists, we write

$$R \cong S.$$

Definition 0.4 (Prime Ideal)**.** *An ideal P in a commutative ring R is called a prime ideal if:*

- *$P \neq R$,*
- *whenever $ab \in P$, then*
$$a \in P \quad \text{or} \quad b \in P.$$

Definition 0.5 (Maximal Ideal)**.** *An ideal M in a ring R is called a maximal ideal if:*

- $M \neq R$,
- the only ideals containing M are

$$M \quad \text{and} \quad R.$$

Definition 0.6 (Ideal)**.** *A subset I of a ring R is called an ideal if:*

- $(I, +)$ *is a subgroup of* $(R, +)$,
- *for every* $r \in R$ *and* $a \in I$,

$$ra \in I \quad \text{and} \quad ar \in I.$$

Definition 0.7 (Unit)**.** *An element $a \in R$ is invertible and called a unit if there exists $b \in R$ such that*

$$ab = ba = 1.$$

For example,

$$(a + b\varepsilon)^{-1} = \frac{1}{a} - \frac{b}{a^2}\varepsilon, \quad a \neq 0.$$

Definition 0.8 (Zero Divisor)**.** *An element a in a ring R is called a zero divisor if there exists a nonzero element $b \in R$ such that*

$$ab = 0.$$

Example 0.9. *In the ring of dual numbers,*

$$\varepsilon^2 = 0,$$

while $\varepsilon \neq 0$. Therefore, ε is a zero divisor.

Definition 0.10 (Invertible Element)**.** *An element a in a ring is called invertible (or a unit) if there exists an element a^{-1} such that*

$$aa^{-1} = a^{-1}a = 1.$$

Example 0.11. *For a dual number*

$$z = a + b\varepsilon,$$

the element is invertible if and only if $a \neq 0$. Its inverse is

$$z^{-1} = \frac{1}{a} - \frac{b}{a^2}\varepsilon.$$

Definition 0.12 (Field)**.** *A field is an algebraic structure in which:*

- *addition and multiplication are defined,*
- *every nonzero element has a multiplicative inverse,*
- *distributive laws hold.*

Example 0.13. *The sets*

$$\mathbb{Q}, \quad \mathbb{R}, \quad \mathbb{C}$$

are fields.

The set of dual numbers is not a field because it contains nonzero zero divisors such as ε, which are not invertible.

Definition 0.14 (Nilpotent Element)**.** *An element $a \in R$ is called nilpotent if there exists a positive integer n such that*

$$a^n = 0$$

for some $n \in \mathbb{N}$.

For example,

$$\varepsilon^2 = 0.$$

Definition 0.15 (Idempotent Element)**.** *An element $a \in R$ is idempotent if*

$$a^2 = a.$$

For example,

$$0^2 = 0 \qquad \text{where } 0 \in \mathbb{Z}$$

$$1^2 = 1 \qquad \text{where } 1 \in \mathbb{Z}$$

Definition 0.16 (Vector Space)**.** *Let V be a nonempty set and let F be a field.*

The set V is called a vector space over F if there exist two operations:

1. *vector addition*

$$+ : V \times V \to V$$

2. *scalar multiplication*

$$\cdot : F \times V \to V$$

such that for all $u, v, w \in V$ *and all* $\alpha, \beta \in F$, *the following properties hold:*

1. $u + v = v + u$

2. $(u + v) + w = u + (v + w)$

3. *There exists an element* $0 \in V$ *such that* $v + 0 = v$

4. *For every* $v \in V$, *there exists* $-v \in V$ *such that*

$$v + (-v) = 0$$

5. $\alpha(u + v) = \alpha u + \alpha v$

6. $(\alpha + \beta)v = \alpha v + \beta v$

7. $\alpha(\beta v) = (\alpha\beta)v$

8. $1v = v$

where 1 *denotes the multiplicative identity in* F.

1 ALGEBRAIC STRUCTURE OF DUAL NUMBERS

In this section we examine the algebraic strucre of dual numbers, definitions and theorems

1.1 The Ring of Dual Numbers

Definition of Dual Numbers

Let $\mathbb{R}$ denote the field of real numbers.

The set of dual numbers is defined as

$$\mathbb{D} = \{a + b\varepsilon \mid a, b \in \mathbb{R},\ \varepsilon^2 = 0\},$$

where $\varepsilon \neq 0$ is a formal symbol satisfying $\varepsilon^2 = 0$.

Addition and multiplication on $\mathbb{D}$ are defined as follows:

$$(a + b\varepsilon) + (c + d\varepsilon) = (a + c) + (b + d)\varepsilon,$$

$$(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc)\varepsilon,$$

using the relation $\varepsilon^2 = 0$.

Algebraic Structure

Theorem 1.3

The set $\mathbb{D}$ equipped with the above operations forms a commutative ring with identity.

Proof. We verify the ring axioms.

(1) Closure

Let

$$x = a + b\varepsilon, \qquad y = c + d\varepsilon$$

be elements of $\mathbb{D}$. Then

$$x + y = (a + c) + (b + d)\varepsilon \in \mathbb{D},$$

$$xy = ac + (ad + bc)\varepsilon \in \mathbb{D}.$$

Hence, D is closed under addition and multiplication.

(2) Additive Structure

Addition is associative and commutative since it is inherited from $\mathbb{R}$.

The additive identity is

$$0 = 0 + 0\varepsilon.$$

For any $a + b\varepsilon \in \mathbb{D}$, its additive inverse is

$$-(a + b\varepsilon) = -a - b\varepsilon.$$

Thus, $(\mathbb{D}, +)$ is an abelian group.

(3) Multiplication

Multiplication is associative because it is derived from polynomial multiplication modulo the relation $\varepsilon^2 = 0$.

It is also commutative since

$$(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc)\varepsilon = (c + d\varepsilon)(a + b\varepsilon).$$

(4) Distributive Laws

Let

$$x = a + b\varepsilon, \qquad y = c + d\varepsilon, \qquad z = e + f\varepsilon.$$

Then

$$y + z = (c + e) + (d + f)\varepsilon.$$

Now,

$$x(y + z) = (a + b\varepsilon)((c + e) + (d + f)\varepsilon).$$

Using multiplication in D:

$$x(y + z) = a(c + e) + (a(d + f) + b(c + e))\varepsilon.$$

Expanding:

$$x(y + z) = ac + ae + (ad + af + bc + be)\varepsilon.$$

Also,

$$xy = ac + (ad + bc)\varepsilon,$$

$$xz = ae + (af + be)\varepsilon.$$

Thus,

$$x(y + z) = xy + xz.$$

Now compute the right distributive law.

$$x + y = (a + c) + (b + d)\varepsilon.$$

Then,

$$(x + y)z = ((a + c) + (b + d)\varepsilon)(e + f\varepsilon).$$

So,

$$(x + y)z = (a + c)e + ((a + c)f + (b + d)e)\varepsilon.$$

Expanding:

$$(x + y)z = ae + ce + (af + cf + be + de)\varepsilon.$$

Also,

$$xz = ae + (af + be)\varepsilon,$$

$$yz = ce + (cf + de)\varepsilon.$$

Therefore,

$$(x + y)z = xz + yz.$$

(5) Multiplicative Identity
The element

$$1 = 1 + 0\varepsilon$$

satisfies

$$(1 + 0\varepsilon)(a + b\varepsilon) = a + b\varepsilon,$$

so it is the multiplicative identity.

Therefore, D is a commutative ring with identity. $\square$

Dual Numbers Are Not a Field

Theorem 1.5

The ring D is not a field.

Proof. Observe that

$$\varepsilon \neq 0 \qquad \text{and} \qquad \varepsilon^2 = 0.$$

Thus, ε is a nonzero element whose product with itself is zero. Hence, ε is a zero divisor.

In a field, no nonzero element can be a zero divisor. Therefore, D is not a field. $\square$

Theorem 1.6

The ring $\mathbb{D}$ is an extension of $\mathbb{R}$.

Proof. Each real number $a \in \mathbb{R}$ can be embedded into D via

$$a \mapsto a + 0\varepsilon.$$

Thus,

$$\mathbb{R} \subseteq D.$$

$\square$

Proof. Let $a + b\varepsilon \in \mathbb{D}$. Then

$$a + b\varepsilon = a(1) + b(\varepsilon).$$

Thus, $\mathbb{D}$ is spanned by

$$\{1, \varepsilon\}.$$

Hence,

$$\dim_{\mathbb{R}}(\mathbb{D}) = 2.$$

$\square$

Now, we charactarize some of the special elements in the ring of dual numbers

1.8 Ideal Structure of Dual Numbers

The set of dual numbers

$$\mathbb{D} = \{a + b\varepsilon \mid a, b \in \mathbb{R},\ \varepsilon^2 = 0\}$$

has a rich ideal structure because it contains nilpotent and noninvertible elements.

In the ring of dual numbers, the subset

$$(\varepsilon) = \{b\varepsilon \mid b \in \mathbb{R}\}$$

forms an ideal generated by ε. Since

$$\varepsilon^2 = 0,$$

every element of this ideal is nilpotent.

Moreover, every noninvertible element of $\mathbb{D}$ belongs to the ideal (ε). Therefore, the ideal structure of dual numbers is closely related to the nilpotent behavior of ε.

Prime Ideal Structure

The ideal

$$(\varepsilon)$$

is a prime ideal in $\mathbb{D}$.

Indeed, if

$$xy \in (\varepsilon),$$

then the real part of xy must be zero. Since multiplication of dual numbers satisfies

$$(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc)\varepsilon,$$

we obtain

$$ac = 0.$$

Because $\mathbb{R}$ is a field, this implies

$$a = 0 \quad \text{or} \quad c = 0.$$

Hence,

$$x \in (\varepsilon) \quad \text{or} \quad y \in (\varepsilon).$$

Therefore, (ε) is a prime ideal.

Maximal Ideal Structure

The ideal

$$(\varepsilon)$$

is also a maximal ideal of the ring of dual numbers.

Consider the quotient ring

$$\mathbb{D}/(\varepsilon).$$

Every dual number

$$a + b\varepsilon$$

is identified with its real part a. Hence,

$$\mathbb{D}/(\varepsilon) \cong \mathbb{R}.$$

Since $\mathbb{R}$ is a field, the quotient ring is a field. Therefore, by definition,

$$(\varepsilon)$$

is a maximal ideal.

Consequently, the ideal generated by ε is both prime and maximal.

1.9 Units in $\mathbb{D}$

Theorem 1.10

An element $a + b\varepsilon \in D$ is invertible if and only if $a \neq 0$.

In this case,

$$(a + b\varepsilon)^{-1} = \frac{1}{a} - \frac{b}{a^2}\varepsilon.$$

Proof. Let

$$x = a + b\varepsilon.$$

Suppose it has an inverse

$$y = c + d\varepsilon$$

such that

$$xy = 1.$$

Then

$$(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc)\varepsilon = 1 + 0\varepsilon.$$

Equating coefficients gives

$$ac = 1, \qquad ad + bc = 0.$$

From $ac = 1$, we obtain

$$c = \frac{1}{a},$$

hence $a \neq 0$.

Substituting into the second equation:

$$ad + b\left(\frac{1}{a}\right) = 0$$

which implies

$$d = -\frac{b}{a^2}.$$

Thus,

$$x^{-1} = \frac{1}{a} - \frac{b}{a^2}\varepsilon.$$

Conversely, if $a = 0$, then

$$x = b\varepsilon$$

and

$$x^2 = 0,$$

so x cannot have a multiplicative inverse.

Therefore, $a + b\varepsilon$ is invertible if and only if $a \neq 0$. □

1.11 Nilpotent Elements in $\mathbb{D}$

Theorem 1.12

An element $a + b\varepsilon \in \mathbb{D}$ is nilpotent if and only if $a = 0$.

Proof. Let

$$x = a + b\varepsilon \in \mathbb{D}.$$

We compute its square:

$$x^2 = (a + b\varepsilon)^2.$$

Using distributivity,

$$(a + b\varepsilon)^2 = (a + b\varepsilon)(a + b\varepsilon).$$

Expanding,

$$= a^2 + ab\varepsilon + ba\varepsilon + b^2\varepsilon^2.$$

Since multiplication in $\mathbb{R}$ is commutative, $ab = ba$, so

$$= a^2 + 2ab\varepsilon + b^2\varepsilon^2.$$

Using the defining property of dual numbers,

$$\varepsilon^2 = 0,$$

hence

$$b^2\varepsilon^2 = 0.$$

Thus,

$$x^2 = a^2 + 2ab\varepsilon.$$

If x is nilpotent, then there exists $n \in \mathbb{N}$ such that

$$x^n = 0.$$

In particular, the real part must be zero, which implies

$$a = 0.$$

Conversely, if $a = 0$, then

$$x = b\varepsilon,$$

and

$$x^2 = b^2\varepsilon^2 = 0.$$

Therefore, $a + b\varepsilon$ is nilpotent if and only if $a = 0$. $\square$

Proposition 1.13

The set $\mathbb{D}$ of dual numbers forms an abelian group under addition.

Proof. Let

$$x = a + b\varepsilon, \qquad y = c + d\varepsilon$$

be arbitrary elements of $\mathbb{D}$.
Their sum is

$$x + y = (a + c) + (b + d)\varepsilon.$$

Since addition in $\mathbb{R}$ is commutative,

$$a + c = c + a, \qquad b + d = d + b.$$

Therefore,

$$x + y = (c + a) + (d + b)\varepsilon = y + x.$$

Thus addition in $\mathbb{D}$ is commutative.
Moreover:

- additive identity exists:
$$0 = 0 + 0\varepsilon$$
- additive inverse exists:
$$-(a + b\varepsilon) = -a - b\varepsilon$$
- associativity follows from associativity in $\mathbb{R}$.

Hence $(\mathbb{D}, +)$ is an abelian group. □

1.14 Idempotent Elements

Theorem 1.15

The only idempotent elements in $\mathbb{D}$ are 0 and 1.

Proof. Let

$$x = a + b\varepsilon$$

be idempotent, so

$$x^2 = x.$$

Compute:

$$(a + b\varepsilon)^2 = a^2 + 2ab\varepsilon.$$

Equating coefficients with $a + b\varepsilon$, we obtain

$$a^2 = a, \qquad 2ab = b.$$

From $a^2 = a$, we get

$$a(a - 1) = 0,$$

so $a = 0$ or $a = 1$.
If $a = 0$, then $2ab = 0 = b$, hence $b = 0$.
If $a = 1$, then

$$2b = b$$

which implies $b = 0$.
Thus, the only idempotent elements are 0 and 1. □

1.16 Taylor Expansion in Dual Numbers

Theorem 1.17

Let f be differentiable at $a \in \mathbb{R}$. Then

$$f(a + b\varepsilon) = f(a) + bf'(a)\varepsilon.$$

Proof. Using Taylor expansion,

$$f(a + h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + \cdots$$

Let $h = \varepsilon$. Since $\varepsilon^2 = 0$, all higher-order terms vanish:

$$f(a + \varepsilon) = f(a) + f'(a)\varepsilon.$$

More generally, let

$$h = b\varepsilon,$$

where $b \in \mathbb{R}$.

Using Taylor expansion,

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + \cdots$$

Substituting $h = b\varepsilon$, we obtain

$$f(a+b\varepsilon) = f(a) + f'(a)(b\varepsilon) + \frac{f''(a)}{2}(b\varepsilon)^2 + \cdots$$

Since

$$\varepsilon^2 = 0,$$

we also have

$$(b\varepsilon)^2 = b^2\varepsilon^2 = 0.$$

Thus, all higher-order terms vanish and we get

$$f(a+b\varepsilon) = f(a) + bf'(a)\varepsilon.$$

Therefore, the coefficient b simply scales the infinitesimal part of the derivative term.

□

1.2 Applications of Taylor Expansion

1. Square Root

The square root function on dual numbers can be derived directly from the Taylor expansion formula

$$f(a+b\varepsilon) = f(a) + bf'(a)\varepsilon.$$

Take

$$f(x) = \sqrt{x}.$$

Since

$$f'(x) = \frac{1}{2\sqrt{x}},$$

we obtain

$$f(a+b\varepsilon) = \sqrt{a} + b\left(\frac{1}{2\sqrt{a}}\right)\varepsilon.$$

Therefore,

$$\sqrt{a+b\varepsilon} = \sqrt{a} + \frac{b}{2\sqrt{a}}\varepsilon.$$

This computation illustrates the fundamental idea behind dual numbers: the dual component automatically carries derivative information through the Taylor expansion.

For example,

$$\sqrt{4+\varepsilon} = \sqrt{4} + \frac{1}{2\sqrt{4}}\varepsilon = 2 + \frac{1}{4}\varepsilon.$$

2. Absolute Value

Let

$$z = a + b\varepsilon.$$

The absolute value depends only on the real component:

$$|z| = |a|.$$

This can also be interpreted through Taylor expansion. Consider the function

$$f(x) = |x|.$$

For $a \neq 0$, the derivative is

$$f'(a) = \operatorname{sgn}(a).$$

Applying the dual number Taylor formula gives

$$|a + b\varepsilon| = |a| + b\,\operatorname{sgn}(a)\,\varepsilon.$$

However, the infinitesimal part does not affect the magnitude in the usual real sense, so the norm of a dual number is conventionally defined by

$$|z| = |a|.$$

For example,

$$|3 + 5\varepsilon| = 3.$$

3. Binomial Expansion

The generalized power formula for dual numbers follows naturally from Taylor expansion.

Consider the function

$$f(x) = x^n, \qquad n \in \mathbb{R}.$$

Its derivative is

$$f'(x) = nx^{n-1}.$$

Applying the dual number Taylor formula,

$$f(a + b\varepsilon) = f(a) + bf'(a)\varepsilon,$$

we obtain

$$(a + b\varepsilon)^n = a^n + na^{n-1}b\varepsilon.$$

All higher-order terms disappear automatically because

$$\varepsilon^2 = 0.$$

This means that dual numbers naturally keep track of both the value of a function and its derivative at the same time. The real part gives the function value, while the dual part gives the derivative information.

For example,

$$(2 + 3\varepsilon)^4 = 2^4 + 4(2^3)(3)\varepsilon = 16 + 96\varepsilon.$$

2 MATRIX REPRESENTATION OF DUAL NUMBERS

Let

$$f(A) = \begin{pmatrix} a & a^* \\ 0 & a \end{pmatrix}$$

where $a, a^* \in \mathbb{R}$.

If M is the set of matrices of this form, then

$$(M, +, \cdot)$$

is a commutative ring with identity.

Moreover, the rings

$$(M, +, \cdot) \qquad \text{and} \qquad (\mathbb{D}, \oplus, \odot)$$

are isomorphic.

For this reason, some references define the ring of dual numbers directly as the set of matrices

$$\begin{pmatrix} a & a^* \\ 0 & a \end{pmatrix}.$$

Theorem 2.1

Let $a, a^* \in \mathbb{R}$.

If M denotes the set of matrices of the form

$$\begin{pmatrix} a & a^* \\ 0 & a \end{pmatrix},$$

then M is isomorphic to the set of dual numbers $\mathbb{D}$.

Proof. Define the function

$$f : \mathbb{D} \to M$$

by

$$f(a, a^*) = \begin{pmatrix} a & a^* \\ 0 & a \end{pmatrix}.$$

We will show that f is an isomorphism.

f is Linear

Let

$$A = (a, a^*), \qquad B = (b, b^*) \in \mathbb{D}$$

and let $\lambda \in \mathbb{R}$.
Then

$$f(A + \lambda B) = f(A) + \lambda f(B).$$

Indeed,

$$A + \lambda B = (a + \lambda b, a^* + \lambda b^*).$$

Therefore,

$$f(A + \lambda B) = \begin{pmatrix} a + \lambda b & a^* + \lambda b^* \\ 0 & a + \lambda b \end{pmatrix}.$$

Also,

$$f(A) + \lambda f(B) = \begin{pmatrix} a & a^* \\ 0 & a \end{pmatrix} + \lambda \begin{pmatrix} b & b^* \\ 0 & b \end{pmatrix}.$$

Thus,

$$f(A + \lambda B) = f(A) + \lambda f(B).$$

*

f Preserves Multiplication
Using the multiplication rule of dual numbers,

$$A \cdot B = (ab, ab^* + a^*b),$$

we obtain

$$f(A \cdot B) = \begin{pmatrix} ab & ab^* + a^*b \\ 0 & ab \end{pmatrix}.$$

On the other hand,

$$\begin{pmatrix} a & a^* \\ 0 & a \end{pmatrix} \begin{pmatrix} b & b^* \\ 0 & b \end{pmatrix} = \begin{pmatrix} ab & ab^* + a^*b \\ 0 & ab \end{pmatrix}.$$

Thus,

$$f(A \cdot B) = f(A)f(B).$$

f is One-to-One

If $A \neq B$, then

$$(a, a^*) \neq (b, b^*).$$

Hence either

$$a \neq b \quad \text{or} \quad a^* \neq b^*.$$

Therefore,

$$f(A) \neq f(B).$$

So f is injective.

f is Onto

Every matrix in M has the form

$$\begin{pmatrix} a & a^* \\ 0 & a \end{pmatrix}$$

for some unique dual number $(a, a^*) \in \mathbb{D}$.

Hence every element of M is the image of an element in $\mathbb{D}$.

So f is surjective.

Since f is bijective and preserves addition and multiplication, $\mathbb{D}$ and M are isomorphic.

□

2.1 Important Elements in the Matrix Representation

Additive Identity

The additive identity corresponds to the dual number $(0, 0)$:

$$f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Multiplicative Identity

The multiplicative identity corresponds to the dual number $(1, 0)$:

$$f(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Nilpotent Element

The dual unit

$$\varepsilon = (0, 1)$$

corresponds to

$$f(0,1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since

$$\varepsilon^2 = 0,$$

this matrix is nilpotent:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Idempotent Elements

An element A is called idempotent if

$$A^2 = A.$$

Consider the matrix representation of a dual number

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix}.$$

For $A^2 = A$, we must have

$$a^2 = a$$

and

$$2ab = b.$$

From

$$a^2 = a,$$

we obtain

$$a = 0 \quad \text{or} \quad a = 1.$$

If $a = 0$, then

$$0 = b,$$

so the zero matrix is idempotent.

If $a = 1$, then

$$2b = b,$$

which implies

$$b = 0.$$

Therefore, the idempotent elements are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, the only idempotent elements are 0 and 1.

Inverse Elements

Consider the matrix

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

The determinant of A is

$$\det(A) = a^2.$$

Therefore, A is invertible if and only if

$$a \neq 0.$$

Assume $a \neq 0$. Then the inverse matrix is

$$A^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a^2} \\ 0 & \frac{1}{a} \end{pmatrix}.$$

Indeed,

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} \frac{1}{a} & -\frac{b}{a^2} \\ 0 & \frac{1}{a} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, a dual-number matrix has an inverse exactly when its real part is nonzero.

Zero Divisors

A nonzero matrix A is called a zero divisor if there exists a nonzero matrix B such that

$$AB = 0.$$

Consider the matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$N^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$N \neq 0 \quad \text{but} \quad N^2 = 0.$$

Hence, N is a zero divisor.

This corresponds to the dual number ε, which satisfies

$$\varepsilon^2 = 0.$$

3 FUNCTIONS OF DUAL NUMBERS

3.1 Dual Functions

Definition 3.1

Let

$$\mathbb{D} = \{a + b\varepsilon \mid a, b \in \mathbb{R},\ \varepsilon^2 = 0\}.$$

A function

$$f : \mathbb{D} \to \mathbb{D}$$

is called a dual function.

If

$$z = a + b\varepsilon$$

and f is differentiable, then

$$f(a + b\varepsilon) = f(a) + bf'(a)\varepsilon.$$

3.2 Derivative of Dual Functions

Theorem 3.2

Let f be differentiable at $a \in \mathbb{R}$. Then

$$f(a + b\varepsilon) = f(a) + bf'(a)\varepsilon.$$

Proof. Using Taylor expansion,

$$f(a + h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + \cdots$$

Let

$$h = b\varepsilon.$$

Since

$$\varepsilon^2 = 0,$$

we obtain

$$h^2 = 0.$$

Hence,

$$f(a + b\varepsilon) = f(a) + bf'(a)\varepsilon.$$

□

3.3 Trigonometric Dual Functions

Theorem 3.3

For every dual number

$$z = a + b\varepsilon,$$

we have

$$\sin(a + b\varepsilon) = \sin a + b\cos a\,\varepsilon.$$

Proof. Using Taylor expansion,

$$\sin(a + h) = \sin a + h\cos a + \cdots$$

with

$$h = b\varepsilon.$$

Since

$$\varepsilon^2 = 0,$$

higher-order terms vanish. □

Theorem 3.4

For every dual number

$$z = a + b\varepsilon,$$

we have

$$\cos(a + b\varepsilon) = \cos a - b\sin a\,\varepsilon.$$

Proof. Using Taylor expansion,

$$\cos(a + h) = \cos a - h\sin a + \cdots$$

with

$$h = b\varepsilon.$$

Since

$$\varepsilon^2 = 0,$$

higher-order terms vanish. □

Theorem 3.5

For every dual number

$$z = a + b\varepsilon,$$

we have

$$\tan(a + b\varepsilon) = \tan a + b\sec^2(a)\varepsilon.$$

Proof. Apply the derivative formula to

$$f(x) = \tan x.$$

Since

$$f'(x) = \sec^2 x,$$

the result follows. □

Examples

Example 3.6.

$$\sin\left(\frac{\pi}{2} + \varepsilon\right) = 1.$$

Example 3.7.

$$\cos(0 + 3\varepsilon) = 1.$$

Example 3.8.

$$\tan(1 + 2\varepsilon) = \tan(1) + 2\sec^2(1)\varepsilon.$$

3.4 Exponential and Logarithmic Functions

Theorem 3.6

For every dual number

$$z = a + b\varepsilon,$$

we have

$$e^{a+b\varepsilon} = e^a(1 + b\varepsilon).$$

Proof. Using Taylor expansion,

$$e^{a+h} = e^a(1 + h + \cdots)$$

with

$$h = b\varepsilon.$$

Since

$$\varepsilon^2 = 0,$$

higher-order terms vanish. □

Theorem 3.7

For $a > 0$,

$$\ln(a + b\varepsilon) = \ln a + \frac{b}{a}\varepsilon.$$

Proof. Apply the derivative formula to

$$f(x) = \ln x.$$

Since

$$f'(x) = \frac{1}{x},$$

the result follows. □

4 APPLICATIONS IN AUTOMATIC DIFFERENTIATION

4.1 Automatic Differentiation

Automatic differentiation is different from symbolic differentiation and numerical differentiation.

Symbolic differentiation attempts to convert a computer program into a single mathematical expression. This process may produce very complicated and inefficient formulas.

Numerical differentiation, such as the finite difference method, approximates derivatives using small changes in the input values. However, this method may introduce round-off and cancellation errors.

Both classical methods also become inefficient when computing higher-order derivatives or partial derivatives with respect to many variables.

Automatic differentiation avoids these difficulties by computing derivatives directly and accurately during the evaluation process.

4.2 Applications

Due to its efficiency and accuracy, automatic differentiation has many applications in scientific computing and mathematics.

There are several computational implementations of automatic differentiation, including:

INTLAB, Sollya, InCLosure.

In practice, automatic differentiation is generally divided into two modes:

- Forward mode

- Reverse mode

These methods are widely used in:

- nonlinear optimization,
- sensitivity analysis,
- robotics,
- machine learning,
- computer graphics,
- computer vision.

Automatic differentiation is especially important in machine learning because it allows efficient computation of gradients and enables backpropagation in neural networks without manually calculating derivatives.

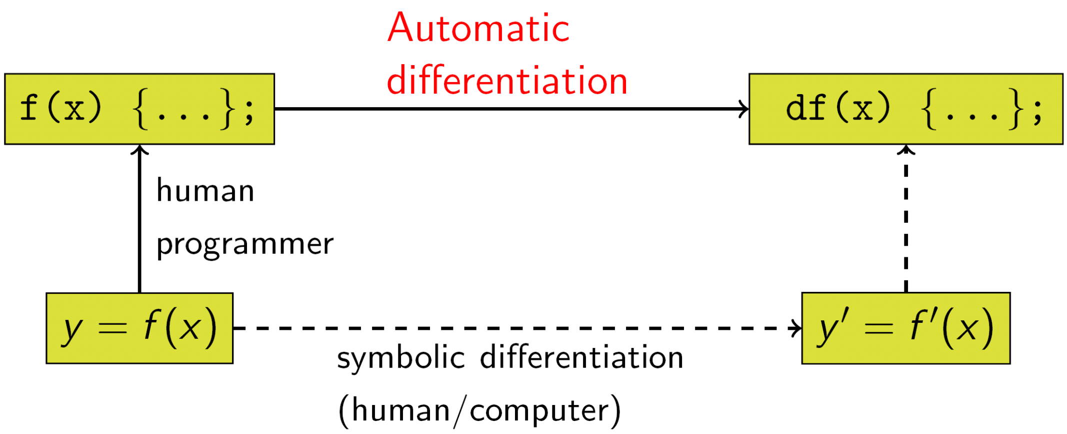

Figure 1: Automatic Differentiation Diagram

Suppose we evaluate

$$f(x) = x^2$$

at

$$x = a + \varepsilon.$$

Then

$$(a + \varepsilon)^2 = a^2 + 2a\varepsilon.$$

The coefficient of ε gives the derivative:

$$f'(a) = 2a.$$

This is the central mechanism behind forward-mode automatic differentiation.

Comparison with Complex Numbers

Property	Complex Numbers	Dual Numbers
Special element	$i^2 = -1$	$\varepsilon^2 = 0$
Norm behavior	Positive definite	non-Standard
Zero divisors	None	Exist
Geometry	Rotations	Very Small Transformations

5 Conclusion

Dual numbers provide a simple yet powerful algebraic system where infinitesimal quantities can be manipulated rigorously.

Because $\varepsilon^2 = 0$, they naturally encode first-order derivatives and therefore play an important role in automatic differentiation, differential geometry, and kinematics.

References

[1] E. Study, *Geometrie der Dynamen*, B. G. Teubner, Leipzig, 1903.

[2] G. Sobczyk, *The Hyperbolic Number Plane*, The College Mathematics Journal, Vol. 26, No. 4, 1995, pp. 268–280.

[3] S. Gull, A. Lasenby, C. Doran, *Imaginary Numbers are not Real — the Geometric Algebra of Spacetime*, Foundations of Physics, Vol. 23, No. 9, 1993, pp. 1175–1201.

[4] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Edition, Kluwer Academic Publishers, 1999.

[5] S. Roman, *Advanced Linear Algebra*, 3rd Edition, Springer, 2008.

[6] S. Lang, *Algebra*, Revised 3rd Edition, Springer, 2002.

[7] D. S. Dummit and R. M. Foote, *Abstract Algebra*, 3rd Edition, Wiley, 2004.

[8] M. Spivak, *Calculus*, 4th Edition, Publish or Perish, 2008.

[9] W. Rudin, *Principles of Mathematical Analysis*, 3rd Edition, McGraw–Hill, 1976.

[10] A. Griewank and A. Walther, *Evaluating Derivatives: Principles and Techniques of Algorithmic Differentiation*, 2nd Edition, SIAM, 2008.

[11] A. G. Baydin, B. A. Pearlmutter, A. A. Radul, J. M. Siskind, *Automatic Differentiation in Machine Learning: a Survey*, Journal of Machine Learning Research, Vol. 18, 2018, pp. 1–43.

[12] R. D. Neidinger, *Introduction to Automatic Differentiation and MATLAB Object-Oriented Programming*, SIAM Review, Vol. 52, No. 3, 2010, pp. 545–563.

[13] K. T. McDonald, *Dual Numbers and Their Applications in Mechanics*, Princeton University Lecture Notes, 2005.

[14] J. M. Selig, *Geometric Fundamentals of Robotics*, 2nd Edition, Springer, 2005.

[15] B. C. Hall, *Lie Groups, Lie Algebras, and Representations*, 2nd Edition, Springer, 2015.

[16] H. Anton and C. Rorres, *Elementary Linear Algebra*, 11th Edition, Wiley, 2013.

[17] G. Strang, *Linear Algebra and Its Applications*, 4th Edition, Brooks/Cole, 2006.