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Graduation Project

Discrete and Continous Boundary Problems

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1 Introduction

The study of boundary problems is a fundamental aspect of mathematical analysis, with applications spanning physics, engineering, and computational science. These problems arise when determining solutions to differential or difference equations that satisfy specific conditions at the boundaries of a domain. Broadly, boundary problems can be classified into two categories: discrete and continuous. Discrete boundary problems involve systems described by recurrence relations, whereas continuous boundary problems are governed by differential equations.

In discrete boundary problems, the system consists of a finite or infinite sequence of entities interacting according to specified rules. These problems are often represented by recurrence relations, which provide powerful tools for understanding the behavior of systems with discrete structures. On the other hand, continuous boundary problems involve differential equations that describe systems with continuous variables, often representing smooth transitions in physical or mathematical contexts. Despite their apparent differences, discrete and continuous boundary problems share profound mathematical connections, and many techniques developed for one domain find analogs in the other.

One of the key goals in the study of boundary problems is to explore these connections and develop unified frameworks that incorporate both discrete and continuous perspectives. For example, the behavior of a vibrating string can be modeled using a differential equation when the string's density is continuous or by a recurrence relation if the string is modeled as a series of discrete masses. In both cases, boundary conditions, such as fixed ends or free vibrations, play a crucial role in determining the system's behavior.

A central theme in this field is the exploration of spectral properties, which involve analyzing the eigenvalues and eigenfunctions associated with boundary problems. These properties reveal critical insights into the stability, oscillatory behavior, and resonance phenomena of the system. The interplay between discrete and continuous methods becomes especially important in spectral analysis, as many discrete problems serve as approximations to their continuous counterparts, and vice versa.

The primary focus of this project lies in the analysis of boundary problems for discrete systems, specifically those that serve as discrete analogs of classical Sturm-Liouville theory. This theory, central to the study of boundary problems, deals with second-order linear differential equations and their associated eigenvalues and eigenfunctions. The discrete counterpart extends this framework to recurrence relations, emphasizing their oscillatory properties, orthogonality conditions, and spectral functions.

In the discrete setting, recurrence relations form the foundation for understanding boundary problems. These relations, often expressed in the form:

$$y_{n+1} - 2y_n + y_{n-1} + \lambda w_n y_n = 0, \tag{1}$$

where y_n represents the sequence of interest, w_n is a weight function, and λ is the spectral parameter, provide a structured way to study the system's behavior. Such relations encapsulate the discrete analogs of continuous differential equations and allow for detailed spectral analysis.

One significant aspect of this analysis involves investigating the oscillatory nature of solutions. The number of zeros of the solutions, or their oscillation count, is directly related to the corresponding eigenvalues, a property mirroring the behavior in continuous Sturm-Liouville theory. Additionally, the orthogonality of eigenfunctions plays a critical role, enabling the expansion of arbitrary sequences in terms of the system's eigenfunctions, akin to Fourier series in the continuous domain.

Spectral functions further enrich this study by providing a comprehensive description of the system's behavior. These functions, which encapsulate the distribution of eigenvalues, are crucial for understanding the resonance and stability properties of the system. The analysis of spectral functions for discrete systems not only parallels that of continuous systems but also highlights unique features that arise in the discrete framework.

This project delves deeply into the mathematical properties and applications of these discrete boundary problems. By building on the foundational principles of Sturm-Liouville theory, it seeks to uncover possibly new insights and extend existing methods to tackle more complex scenarios. The discrete perspective, while inspired by continuous analogs, provides a distinct approach that enriches the overall understanding of boundary problems and their spectral properties.

2 Finite Orthogonal Polynomials

2.1 The Recurrence Relation

We take up here boundary problems of Sturm-Liouville type associated with the recurrence formula

$$c_n y_{n+1} = (a_n \lambda + b_n) y_n - c_{n-1} y_{n-1}, \quad n = 0, \dots, m-1, \tag{2.1.1}$$

where the a_n, b_n, and c_n are real scalars, subject to

$$a_n > 0, \quad c_n > 0. \tag{2.1.2}$$

A boundary problem is given if we ask for sequences $y_{-1}, \dots, y_m$ connected by this relation, not all zero, and satisfying the boundary conditions

$$y_{-1} = 0, \quad y_m + h y_{m-1} = 0, \tag{2.1.3}$$

where h is some fixed real number.

That this is a problem of eigenvalue type, soluble only for isolated values of λ, is easily seen if we construct a typical solution, that is to say, a sequence satisfying (2.1.1) and the first of the boundary conditions (2.1.3), and not vanishing throughout. We must, of course, take $y_0 \neq 0$, since otherwise by (2.1.1) $y_1 = 0, y_2 = 0, \dots$, and the sequence vanishes identically. It will be convenient to define a standard solution

$$y_{-1}(\lambda), y_0(\lambda), y_1(\lambda), \dots, y_m(\lambda), \tag{2.1.4}$$

of (2.1.1) with the fixed initial conditions

$$y_{-1}(\lambda) = 0, \quad y_0(\lambda) = 1/c_{-1} > 0. \tag{2.1.5}$$

Now that we have fixed $y_{-1}(\lambda), y_0(\lambda)$, the values of $y_1(\lambda), y_2(\lambda), \dots$ are to be found successively from (2.1.1). For $n \geq 0$, it is evident that $y_n(\lambda)$ is a polynomial of degree precisely n. We can now say that the remaining boundary condition in (2.1.3) will be satisfied if

$$y_m(\lambda) + h y_{m-1}(\lambda) = 0. \tag{2.1.6}$$

The roots of this equation, the eigenvalues, are thus the zeros of a polynomial of degree m. For if (2.1.6) holds, the sequence (2.1.4) certainly satisfies the conditions (2.1.1), (2.1.3) of the boundary problem, without vanishing identically; conversely, it is easy to prove that any solution of (2.1.1) and (2.1.3), not vanishing identically, must be a sequence proportional to (2.1.4) for such a λ-value.

In showing that the eigenvalues of our boundary problem are the zeros of certain polynomials we begin to approach the theory of orthogonal polynomials. It is not immediately apparent that the polynomials (2.1.4) defined by (2.1.1) and (2.1.5) have any orthogonality properties.We are here considering only the orthogonality of finite sets (2.1.4).

2.2 Lagrange-Type Identities

We have first the following theorem.

Theorem 2.1.1. For $0 \leq n < m$,

$$(\lambda - \mu) \sum_{n=0}^{m} a_n y_n(\lambda) y_n(\mu) = c_n \begin{vmatrix} y_{n+1}(\lambda) & y_{n+1}(\mu) \\ y_n(\lambda) & y_n(\mu) \end{vmatrix}. \tag{2.2.1}$$

Proof: The proof is by induction. We write the recurrence relation for the two arguments in full, giving

$$\begin{aligned} c_n y_{n+1}(\lambda) &= (a_n \lambda + b_n) y_n(\lambda) - c_{n-1} y_{n-1}(\lambda), \\ c_n y_{n+1}(\mu) &= (a_n \mu + b_n) y_n(\mu) - c_{n-1} y_{n-1}(\mu). \end{aligned}$$

Subtracting these two equations, we have

$$\begin{aligned} c_n\big(y_{n+1}(\lambda) y_n(\mu) - y_{n+1}(\mu) y_n(\lambda)\big) &= a_n(\lambda - \mu) y_n(\lambda) y_n(\mu) \\ &\quad + b_n\big(y_n(\lambda) y_n(\mu) - y_n(\mu) y_n(\lambda)\big) \\ &\quad + c_{n-1}\big(y_{n-1}(\mu) y_n(\lambda) - y_{n-1}(\lambda) y_n(\mu)\big). \end{aligned}$$

Simplifying, we obtain

$$\begin{aligned} c_n\big(y_{n+1}(\lambda) y_n(\mu) - y_{n+1}(\mu) y_n(\lambda)\big) &= a_n(\lambda - \mu) y_n(\lambda) y_n(\mu) \\ &\quad + c_{n-1}\big(y_n(\lambda) y_{n-1}(\mu) - y_n(\mu) y_{n-1}(\lambda)\big). \end{aligned}$$

(2.2.2)

Now, let $n = 0$ and recalling that $y_{-1}(\lambda) = y_{-1}(\mu) = 0$, we derive (2.2.1) with $n = 0$. We're going to use induction over n then yields (2.2.1) from (2.2.2) in the general case. We aim to prove the identity

$$c_n\big(y_{n+1}(\lambda) y_n(\mu) - y_{n+1}(\mu) y_n(\lambda)\big) = (\lambda - \mu) \sum_{k=0}^{n} a_k y_k(\lambda) y_k(\mu).$$

This will be done using mathematical induction.

Base Case $(n = 0)$

For $n = 0$, the identity becomes

$$c_0\big(y_1(\lambda)y_0(\mu) - y_1(\mu)y_0(\lambda)\big) = (\lambda - \mu)a_0y_0(\lambda)y_0(\mu).$$

From the recurrence relation, we get

$$c_0y_1(\lambda) = (a_0\lambda + b_0)y_0(\lambda), \quad c_0y_1(\mu) = (a_0\mu + b_0)y_0(\mu).$$

We substitute this into the determinant

$$c_0\big((a_0\lambda + b_0)y_0(\lambda)y_0(\mu) - (a_0\mu + b_0)y_0(\mu)y_0(\lambda)\big).$$

Expanding and simplifying the expression

$$c_0\big(y_1(\lambda)y_0(\mu) - y_1(\mu)y_0(\lambda)\big) = c_0\big(a_0(\lambda - \mu)y_0(\lambda)y_0(\mu)\big).$$

Canceling c_0 then we have

$$(\lambda - \mu)a_0y_0(\lambda)y_0(\mu).$$

Thus, the base case is verified.

Induction Hypothesis

Assume that the identity holds for $n = m$

$$c_m\big(y_{m+1}(\lambda)y_m(\mu) - y_{m+1}(\mu)y_m(\lambda)\big) = (\lambda - \mu)\sum_{k=0}^{m} a_ky_k(\lambda)y_k(\mu).$$

Induction Step $(n = m + 1)$

We now prove that the identity holds for $n = m + 1$. For $n = m + 1$, the left-hand side is

$$c_{m+1}\big(y_{m+2}(\lambda)y_{m+1}(\mu) - y_{m+2}(\mu)y_{m+1}(\lambda)\big).$$

By using the recurrence relation, we have two expressions

$$c_{m+1}y_{m+2}(\lambda) = (a_{m+1}\lambda + b_{m+1})y_{m+1}(\lambda) - c_my_m(\lambda),$$

$$c_{m+1}y_{m+2}(\mu) = (a_{m+1}\mu + b_{m+1})y_{m+1}(\mu) - c_my_m(\mu).$$

Substituting these into the determinant, we get

$$\begin{gathered} c_{m+1}\big(y_{m+2}(\lambda)y_{m+1}(\mu) - y_{m+2}(\mu)y_{m+1}(\lambda)\big) \\ = \big((a_{m+1}\lambda + b_{m+1})y_{m+1}(\lambda)y_{m+1}(\mu) - (a_{m+1}\mu + b_{m+1})y_{m+1}(\mu)y_{m+1}(\lambda)\big) \\ -c_m\big(y_m(\mu)y_{m+1}(\lambda) - y_m(\lambda)y_{m+1}(\mu)\big). \end{gathered}$$

Simplifying the first term, we have

$$\big((a_{m+1}\lambda + b_{m+1})y_{m+1}(\lambda)y_{m+1}(\mu) - (a_{m+1}\mu + b_{m+1})y_{m+1}(\mu)y_{m+1}(\lambda)\big).$$

This simplifies to the following

$$(a_{m+1}(\lambda - \mu))y_{m+1}(\lambda)y_{m+1}(\mu).$$

For the second term, by using the induction hypothesis, we obtain

$$-c_m\big(y_m(\mu)y_{m+1}(\lambda) - y_m(\lambda)y_{m+1}(\mu)\big) = -(\lambda - \mu)\sum_{k=0}^{m} a_k y_k(\lambda)y_k(\mu).$$

Combining both terms, we get

$$\begin{aligned} c_{m+1}\big(y_{m+2}(\lambda)y_{m+1}(\mu) - y_{m+2}(\mu)y_{m+1}(\lambda)\big) = (\lambda - \mu)\big(&a_{m+1}y_{m+1}(\lambda)y_{m+1}(\mu) \\ &+ \sum_{k=0}^{m} a_k y_k(\lambda)y_k(\mu)\big). \end{aligned}$$

Expanding the summation, we obtain

$$c_{m+1}\big(y_{m+2}(\lambda)y_{m+1}(\mu) - y_{m+2}(\mu)y_{m+1}(\lambda)\big) = (\lambda - \mu)\sum_{k=0}^{m+1} a_k y_k(\lambda)y_k(\mu).$$

Thus, the induction step is complete. Therefore, by the principle of mathematical induction, the identity

$$c_n\big(y_{n+1}(\lambda)y_n(\mu) - y_{n+1}(\mu)y_n(\lambda)\big) = (\lambda - \mu)\sum_{k=0}^{n} a_k y_k(\lambda)y_k(\mu)$$

is valid for all $n \geq 0$. ■

We can deduce two important special cases by using the identities below

Starting with the identity, we get the following

$$c_n\big(y_{n+1}(\lambda)y_n(\mu) - y_{n+1}(\mu)y_n(\lambda)\big) = (\lambda - \mu)\sum_{k=0}^{n} a_k y_k(\lambda)y_k(\mu). \qquad (2.2.1)$$

We divide both sides by $(\lambda - \mu)$ to get that equality

$$\frac{c_n\big(y_{n+1}(\lambda)y_n(\mu) - y_{n+1}(\mu)y_n(\lambda)\big)}{\lambda - \mu} = \sum_{k=0}^{n} a_k y_k(\lambda)y_k(\mu).$$

Now, we take the limit as $\mu \to \lambda$. On the left-hand side, we observe that both the numerator and the denominator vanish as $\mu \to \lambda$, so we apply L'Hopital's Rule. Differentiating with respect to μ, we obtain

$$\lim_{\mu\to\lambda} \frac{c_n\big(y_{n+1}(\lambda)y_n(\mu) - y_{n+1}(\mu)y_n(\lambda)\big)}{\lambda-\mu} = c_n\big(y_{n+1}(\lambda)y_n'(\lambda) - y_{n+1}'(\lambda)y_n(\lambda)\big).$$

On the right-hand side, taking the limit as $\mu \to \lambda$ simplifies to

$$\lim_{\mu\to\lambda} \sum_{k=0}^{n} a_k y_k(\lambda) y_k(\mu) = \sum_{k=0}^{n} a_k y_k(\lambda)^2.$$

Thus, the resulting identity is

$$c_n\big(y_{n+1}(\lambda)y_n'(\lambda) - y_{n+1}'(\lambda)y_n(\lambda)\big) = \sum_{k=0}^{n} a_k y_k(\lambda)^2.$$

Theorem 2.2.2. For $0 \le n < m$,

$$\sum_{m=0}^{n} a_m\big(y_m(\lambda)\big)^2 = c_n \begin{vmatrix} y_{n+1}(\lambda) & y_{n+1}'(\lambda) \\ y_n(\lambda) & y_n'(\lambda) \end{vmatrix}. \tag{2.2.3}$$

In particular, for real λ,

$$y_{n+1}'(\lambda)y_n(\lambda) - y_{n+1}(\lambda)y_n'(\lambda) > 0. \tag{2.2.4}$$

Proof:

We aim to prove the inequality

$$y_{n+1}'(\lambda)y_n(\lambda) - y_{n+1}(\lambda)y_n'(\lambda) > 0.$$

This inequality is derived from (2.2.3)

$$\sum_{m=0}^{n} a_m\big(y_m(\lambda)\big)^2 = c_n\big(y_{n+1}(\lambda)y_n'(\lambda) - y_{n+1}'(\lambda)y_n(\lambda)\big).$$

On the left-hand side, we have the summation

$$\sum_{m=0}^{n} a_m\big(y_m(\lambda)\big)^2.$$

Each term in the summation satisfies the following : $a_n > 0$, since the weight coefficients are positive, $\big(y_n(\lambda)\big)^2 \ge 0$, because the square of any term is non-negative.

Thus, the summation is strictly positive

$$\sum_{m=0}^{n} a_m\big(y_m(\lambda)\big)^2 > 0.$$

On the right-hand side of the identity, the term c_n is multiplied by the determinant

$$c_n\big(y_{n+1}(\lambda)y_n'(\lambda) - y_{n+1}'(\lambda)y_n(\lambda)\big).$$

Since $c_n > 0$, the positivity of the determinant determines the positivity of the right-hand side. By using the identity we get

$$\sum_{m=0}^{n} a_m\big(y_m(\lambda)\big)^2 = c_n\big(y_{n+1}(\lambda)y_n'(\lambda) - y_{n+1}'(\lambda)y_n(\lambda)\big),$$

We have already established that

$$\text{The left-hand side is positive } (> 0) \quad \text{and} \quad c_n > 0.$$

This implies that the determinant must also be positive

$$y_{n+1}'(\lambda)y_n(\lambda) - y_{n+1}(\lambda)y_n'(\lambda) > 0.$$

Therefore, we have proven the inequality

$$y_{n+1}'(\lambda)y_n(\lambda) - y_{n+1}(\lambda)y_n'(\lambda) > 0.$$

■

Theorem 2.2.3. For $0 \le n < m$, and complex λ,

$$\sum_{m=0}^{n} a_m|y_m(\lambda)|^2 = \big(2i\,\mathrm{Im}\,\lambda\big)^{-1} c_n \begin{vmatrix} y_{n+1}(\lambda) & y_n(\lambda) \\ y_{n+1}(\lambda) & y_n(\lambda) \end{vmatrix}.$$

Proof:

We start from the identity given in Theorem 2.2.1

$$c_n\big(y_{n+1}(\lambda)y_n(\mu) - y_{n+1}(\mu)y_n(\lambda)\big) = (\lambda - \mu)\sum_{k=0}^{n} a_k y_k(\lambda)y_k(\mu). \qquad (2.2.1)$$

For Theorem 2.2.3, we consider the special case where $\mu = \overline{\lambda}$ (the complex conjugate of λ). Substituting $\mu = \overline{\lambda}$ into (2.2.1), we have

$$\lambda - \mu = \lambda - \overline{\lambda} = 2i\,\mathrm{Im}\,\lambda.$$

Substituting this back into (2.2.1), we get

$$c_n\big(y_{n+1}(\lambda)y_n(\overline{\lambda}) - y_{n+1}(\overline{\lambda})y_n(\lambda)\big) = (2i\,\mathrm{Im}\,\lambda)\sum_{k=0}^{n} a_k y_k(\lambda)y_k(\overline{\lambda}).$$

Now, using the property of complex conjugates $y_k(\overline{\lambda}) = \overline{y_k(\lambda)}$, the equation becomes

$$c_n\big(y_{n+1}(\lambda)\overline{y_n(\lambda)} - \overline{y_{n+1}(\lambda)}y_n(\lambda)\big) = (2i\,\mathrm{Im}\,\lambda)\sum_{k=0}^{n} a_k|y_k(\lambda)|^2.$$

The term on the left-hand side can be expressed as a determinant

$$y_{n+1}(\lambda)y_n(\bar{\lambda}) - y_{n+1}(\bar{\lambda})y_n(\lambda) = \begin{vmatrix} y_{n+1}(\lambda) & y_n(\lambda) \\ y_{n+1}(\bar{\lambda}) & y_n(\bar{\lambda}) \end{vmatrix}.$$

Thus, we rewrite the equation as

$$c_n \begin{vmatrix} y_{n+1}(\lambda) & y_n(\lambda) \\ y_{n+1}(\bar{\lambda}) & y_n(\bar{\lambda}) \end{vmatrix} = \big(2i \operatorname{Im} \lambda\big) \sum_{k=0}^{n} a_k |y_k(\lambda)|^2.$$

Finally, dividing through by c_n, we obtain

$$\sum_{k=0}^{n} a_k |y_k(\lambda)|^2 = \big(2i \operatorname{Im} \lambda\big)^{-1} c_n \begin{vmatrix} y_{n+1}(\lambda) & y_n(\lambda) \\ \overline{y_{n+1}(\lambda)} & \overline{y_n(\lambda)} \end{vmatrix}.$$

■

We begin with the second standard solution $z_n(\lambda)$, which is defined by the recurrence relation

$$c_n z_{n+1}(\lambda) = (a_n \lambda + b_n) z_n(\lambda) - c_{n-1} z_{n-1}(\lambda), \tag{2.2.7}$$

with the initial conditions

$$z_0(\lambda) = 0, \quad z_{-1}(\lambda) = 1. \tag{2.2.8}$$

For $n \geq 1$, $z_n(\lambda)$ is a polynomial of degree $n-1$. This second solution is independent of $y_n(\lambda)$, forming the foundation for a determinant relation.

Theorem 2.2.4. For $0 \leq n < m$,

$$(\lambda - \mu) \sum_{k=0}^{n} a_k y_k(\lambda) z_k(\mu) = c_n \begin{vmatrix} y_{n+1}(\lambda) & z_{n+1}(\mu) \\ y_n(\lambda) & z_n(\mu) \end{vmatrix} - 1. \tag{2.2.9}$$

Proof:

We start with the identity from Theorem 2.2.4

$$(\lambda - \mu) \sum_{k=0}^{n} a_k y_k(\lambda) z_k(\mu) = c_n \begin{vmatrix} y_{n+1}(\lambda) & z_{n+1}(\mu) \\ y_n(\lambda) & z_n(\mu) \end{vmatrix} - 1. \tag{2.2.9}$$

Now, let $\lambda = \mu$. In this case, the term $(\lambda - \mu)$ on the left-hand side becomes zero, so the equation reduces to

$$0 = c_n \begin{vmatrix} y_{n+1}(\lambda) & z_{n+1}(\lambda) \\ y_n(\lambda) & z_n(\lambda) \end{vmatrix} - 1.$$

Rearranging, we obtain

$$c_n \begin{vmatrix} y_{n+1}(\lambda) & z_{n+1}(\lambda) \\ y_n(\lambda) & z_n(\lambda) \end{vmatrix} = 1.$$

Expanding the determinant, we have

$$\begin{vmatrix} y_{n+1}(\lambda) & z_{n+1}(\lambda) \\ y_n(\lambda) & z_n(\lambda) \end{vmatrix} = y_{n+1}(\lambda)z_n(\lambda) - z_{n+1}(\lambda)y_n(\lambda).$$

Substituting this into the equation, we get

$$c_n\{y_{n+1}(\lambda)z_n(\lambda) - z_{n+1}(\lambda)y_n(\lambda)\} = 1.$$

To prove (2.2.9), we shall start with the recurrence relations

$$c_n y_{n+1}(\lambda) = (a_n\lambda + b_n)y_n(\lambda) - c_{n-1}y_{n-1}(\lambda),$$

$$c_n z_{n+1}(\mu) = (a_n\mu + b_n)z_n(\mu) - c_{n-1}z_{n-1}(\mu).$$

Now, we multiply the first equation by $z_n(\mu)$ and the second equation by $y_n(\lambda)$

$$c_n y_{n+1}(\lambda)z_n(\mu) = (a_n\lambda + b_n)y_n(\lambda)z_n(\mu) - c_{n-1}y_{n-1}(\lambda)z_n(\mu),$$

$$c_n z_{n+1}(\mu)y_n(\lambda) = (a_n\mu + b_n)z_n(\mu)y_n(\lambda) - c_{n-1}z_{n-1}(\mu)y_n(\lambda).$$

Subtracting these two equations, we have

$$c_n\{y_{n+1}(\lambda)z_n(\mu) - z_{n+1}(\mu)y_n(\lambda)\} =$$
$$\big((a_n\lambda + b_n)y_n(\lambda)z_n(\mu) - (a_n\mu + b_n)z_n(\mu)y_n(\lambda)\big) -$$
$$\big(c_{n-1}y_{n-1}(\lambda)z_n(\mu) - c_{n-1}z_{n-1}(\mu)y_n(\lambda)\big).$$

Simplifying the terms, we group the coefficients of $(\lambda - \mu)$ and the remaining terms

$$c_n\{y_{n+1}(\lambda)z_n(\mu) - z_{n+1}(\mu)y_n(\lambda)\} = a_n(\lambda - \mu)y_n(\lambda)z_n(\mu) +$$
$$c_{n-1}\{z_{n-1}(\mu)y_n(\lambda) - y_{n-1}(\lambda)z_n(\mu)\}. \tag{2.2.11}$$

Thus, we have derived the determinant relation (2.2.11). Let us recall the initial conditions and setup for $n = 0$

- $y_{-1}(\lambda) = 0, \quad z_{-1}(\mu) = 1,$
- $y_0(\lambda) = 1, \quad z_0(\mu) = 0,$
- $c_0 = 1.$

For $n = 0$, the summation in (2.2.9) simplifies to a single term, and the determinant reduces to

$$\begin{vmatrix} y_1(\lambda) & z_1(\mu) \\ y_0(\lambda) & z_0(\mu) \end{vmatrix}.$$

Base Case ($n = 0$):

When $n = 0$, the summation on the left-hand side simplifies to

$$(\lambda - \mu)a_0 y_0(\lambda) z_0(\mu).$$

Using the initial conditions, we get

$$y_0(\lambda) = 1, \quad z_0(\mu) = 0, \quad c_0 = 1,$$

and noting that the determinant becomes

$$\begin{vmatrix} y_1(\lambda) & z_1(\mu) \\ y_0(\lambda) & z_0(\mu) \end{vmatrix} = \begin{vmatrix} y_1(\lambda) & z_1(\mu) \\ 1 & 0 \end{vmatrix} = -z_1(\mu),$$

we substitute into the identity then we have

$$(\lambda - \mu) \cdot a_0 \cdot 1 \cdot 0 = 1 \cdot (-z_1(\mu)) - 1.$$

This simplifies to the following

$$0 = -z_1(\mu) - 1 \implies z_1(\mu) = -1.$$

Thus, the base case holds.

Induction Step:

Assume the identity holds for $n = k$:

$$(\lambda - \mu) \sum_{j=0}^{k} a_j y_j(\lambda) z_j(\mu) = c_k \begin{vmatrix} y_{k+1}(\lambda) & z_{k+1}(\mu) \\ y_k(\lambda) & z_k(\mu) \end{vmatrix} - 1.$$

We aim to prove the identity for $n = k + 1$

$$(\lambda - \mu) \sum_{j=0}^{k+1} a_j y_j(\lambda) z_j(\mu) = c_{k+1} \begin{vmatrix} y_{k+2}(\lambda) & z_{k+2}(\mu) \\ y_{k+1}(\lambda) & z_{k+1}(\mu) \end{vmatrix} - 1.$$

Using the induction hypothesis, we separate the summation

$$(\lambda - \mu) \left[\sum_{j=0}^{k} a_j y_j(\lambda) z_j(\mu) + a_{k+1} y_{k+1}(\lambda) z_{k+1}(\mu) \right].$$

At this stage, we use the determinant relation (2.2.11)

$$\begin{aligned} c_{k+1}\{y_{k+2}(\lambda) z_{k+1}(\mu) - z_{k+2}(\mu) y_{k+1}(\lambda)\} = (\lambda - \mu) a_{k+1} y_{k+1}(\lambda) z_{k+1}(\mu) \\ + c_k\{z_k(\mu) y_{k+1}(\lambda) - y_k(\lambda) z_{k+1}(\mu)\}. \end{aligned}$$

Substituting this into the expression, and simplifying, we recover

$$(\lambda - \mu) \sum_{j=0}^{k+1} a_j y_j(\lambda) z_j(\mu) = c_{k+1} \begin{vmatrix} y_{k+2}(\lambda) & z_{k+2}(\mu) \\ y_{k+1}(\lambda) & z_{k+1}(\mu) \end{vmatrix} - 1.$$

Thus, by the principle of mathematical induction, the identity holds for all $n \geq 0$.

■

2.3 Oscillatory Properties

This section focuses on the reality and separation of zeros of the polynomial $y_n(\lambda) + hy_{n-1}(\lambda)$. These properties provide valuable insights into boundary problems like those described in (2.1.1) and (2.1.3). While many techniques exist for analyzing classical polynomials such as Legendre polynomials, we limit our discussion to methods based on recurrence relations and their implications, as outlined in Section 2.2.

Theorem 2.3.1. *For real h, the polynomial*

$$y_n(\lambda) + hy_{n-1}(\lambda) \tag{2.3.1}$$

has precisely n real and simple zeros.

Proof: Suppose if it is possible take λ as a complex zero of (2.3.1). Using this fact and taking also complex conjugates, we have

$$y_n(\lambda) + hy_{n-1}(\lambda) = 0, \quad y_n(\overline{\lambda}) + hy_{n-1}(\overline{\lambda}) = 0. \tag{2.3.2}$$

We then can easily see that the right of (2.2.5) vanishes, firstly recall the recurrence relation

$$c_n y_{n+1}(\lambda) = (a_n\lambda + b_n) y_n(\lambda) - c_{n-1} y_{n-1}(\lambda), \quad a_n > 0, \quad c_n > 0$$

Then focus on the RHS of (2.2.5)

$$\begin{aligned}
\sum_{n=0}^{m} a_n |y_n(\lambda)|^2 &= \frac{c_n}{2\,\mathrm{Im}\,\lambda}\left(y_{n+1}(\lambda)y_n(\overline{\lambda}) - y_n(\lambda)y_{n+1}(\overline{\lambda})\right)\\
&= \frac{1}{2\,\mathrm{Im}\,\lambda}\left(c_n y_{n+1}(\lambda)y_n(\overline{\lambda}) - c_n y_n(\lambda)y_{n+1}(\overline{\lambda})\right)\\
&= \frac{1}{2\,\mathrm{Im}\,\lambda}\Big\{ y_n(\overline{\lambda})\left[(a_n\lambda + b_n)y_n(\lambda) - c_{n-1}y_{n-1}(\lambda)\right]\\
&\quad - y_n(\lambda)\left[(a_n\overline{\lambda} + b_n)y_n(\overline{\lambda}) - c_{n-1}y_{n-1}(\overline{\lambda})\right] \Big\}\\
&= \frac{1}{2\,i\,\mathrm{Im}\,\lambda}\Big\{ (\lambda)a_n|y_n(\lambda)|^2 + b_n|y_n(\lambda)|^2 - c_{n-1}y_{n-1}(\lambda)y_n(\overline{\lambda})\\
&\quad - (\overline{\lambda})a_n|y_n(\lambda)|^2 - b_n|y_n(\lambda)|^2 + c_{n-1}y_{n-1}(\lambda)y_n(\overline{\lambda}) \Big\}\\
&= \frac{1}{2\,i\,\mathrm{Im}\,\lambda}\Big\{ 2\,i\,\mathrm{Im}\,(\lambda)|y_n(\lambda)|^2\\
&\quad + c_{n-1}h|y_{n-1}(\lambda)|^2 - c_{n-1}h|y_{n-1}(\lambda)|^2 \Big\}\\
&= a_n|y_n(\lambda)|^2.
\end{aligned}$$

Since $y_n(\lambda) = 0$, then the RHS of (2.5.2) vanishes. However, this is impossible because the LHS of equation (2.2.5) is

$$\sum_{r=0}^{n} a_r |y_r(\lambda)|^2,$$

where $a_r > 0$ and $|y_r(\lambda)|^2 \geq 0$. Since a_r are positive coefficients and $|y_r(\lambda)|^2$ represents the squared modulus, which is always nonnegative, each term in the summation is nonnegative.

Thus, the entire summation satisfies

$$\sum_{r=0}^{n} a_r |y_r(\lambda)|^2 \geq 0.$$

Therefore, the left-hand side of (2.2.5) is nonnegative. Hence, the zeroes of (2.3.1) are real.

We can conclude that at a hypothetical multiple zero, necessarily real, we should have simultaneously

$$y_n(\lambda) + hy_{n-1}(\lambda) = 0, \quad y_n'(\lambda) + hy_{n-1}'(\lambda) = 0,$$

and so

$$y_n(\lambda)y_{n-1}'(\lambda) - y_n'(\lambda)y_{n-1}(\lambda) = 0,$$

This contradicts to (2.2.4), as shown below The equation derived in the theorem is

$$y_n(\lambda)y_{n-1}'(\lambda) - y_n'(\lambda)y_{n-1}(\lambda) = 0,$$

which implies that the determinant-like combination of $y_n(\lambda)$, $y_n'(\lambda)$, $y_{n-1}(\lambda)$, and $y_{n-1}'(\lambda)$ is zero. This suggests that $y_n(\lambda)$ and $y_{n-1}(\lambda)$ are linearly dependent, meaning there exists a scalar h such that

$$y_n(\lambda) + hy_{n-1}(\lambda) = 0, \quad y_n'(\lambda) + hy_{n-1}'(\lambda) = 0.$$

However, equation (2.2.4) explicitly states that

$$y_n(\lambda)y_{n-1}'(\lambda) - y_n'(\lambda)y_{n-1}(\lambda) = C_n,$$

where $C_n \neq 0$. This means that the determinant is not zero, and $y_n(\lambda)$ and $y_{n-1}(\lambda)$ are linearly independent.

The contradiction arises because if the determinant were zero (as the derivative-based equation suggests), then C_n would also be zero, violating (2.2.4). Hence, the assumption of a multiple zero, which leads to the determinant being zero, is false.

Since (2.3.1), as a polynomial of degree exactly n, must have n zeros altogether, this completes the proof. ■

Theorem 2.3.2. Two consecutive polynomials $y_n(\lambda)$, $y_{n-1}(\lambda)$ have no common zeros. Between any zeros of one of them lies a zero of the other.

Proof: Suppose that λ_1, λ_2 are two zeros of $y_n(\lambda)$, which we take to be consecutive; since $y_n(\lambda)$ has only simple zeros, this implies that $y_n'(\lambda_1)$, $y_n'(\lambda_2)$ have opposite signs.

Using (2.2.4), we show that between any two consecutive roots of $y_n(\lambda)$, there lies a root of $y_{n-1}(\lambda)$. Let λ_1 and λ_2 be two consecutive roots of $y_n(\lambda)$. At these points, we have

$$y_n(\lambda_1) = 0, \quad y_n(\lambda_2) = 0.$$

By (2.2.4), we have

$$y_n(\lambda)y_{n-1}'(\lambda) - y_n'(\lambda)y_{n-1}(\lambda) = K, \quad K \neq 0.$$

Let λ_1 and λ_2 be two consecutive roots of $y_n(\lambda)$. At these points, $y_n(\lambda_1) = 0$ and $y_n(\lambda_2) = 0$. Substituting these conditions into (2.2.4), we get

$$-y_n'(\lambda_1)y_{n-1}(\lambda_1) = K, \quad -y_n'(\lambda_2)y_{n-1}(\lambda_2) = K.$$

Since $K > 0$, it follows that

$$y_n'(\lambda_1)y_{n-1}(\lambda_1) > 0, \quad y_n'(\lambda_2)y_{n-1}(\lambda_2) > 0.$$

Thus, $y_n'(\lambda_1)$ and $y_{n-1}(\lambda_1)$ must have the same sign, and similarly for λ_2. As $y_n(\lambda)$ changes sign between λ_1 and λ_2, we know that $y_n'(\lambda_1)$ and $y_n'(\lambda_2)$ must have opposite signs. Consequently, $y_{n-1}(\lambda_1)$ and $y_{n-1}(\lambda_2)$ must also have opposite signs.

By the Intermediate Value Theorem, $y_{n-1}(\lambda)$ must have a root between λ_1 and λ_2. This proves that between any two consecutive roots of $y_n(\lambda)$, there lies a root of $y_{n-1}(\lambda)$. $n \geq 0$. ■

2.4 Orthogonality

In this section, we focus on orthogonality of eigenfunctions, that is to say, of certain sequences of the form (2.1.4).

Theorem 2.4.1. The sequences

$$y_0(\lambda_r), \ldots, y_{m-1}(\lambda_r), \quad r = 0, \ldots, m-1,$$

are orthogonal according to

$$\sum_{p=0}^{m-1} a_p y_p(\lambda_r) y_p(\lambda_s) = \rho_r \delta_{rs}, \tag{2.4.2}$$

where

$$\rho_r = \sum_{p=0}^{m-1} a_p \{y_p(\lambda_r)\}^2 \tag{2.4.3}$$

and

$$\rho_r = c_{m-1} y_{m-1}(\lambda_r) \big\{ y_m'(\lambda_r) + h y_{m-1}'(\lambda_r) \big\}. \tag{2.4.4}$$

Proof: For the case $r \neq s$, we take $\lambda = \lambda_r$, $\mu = \lambda_s$, $n = m-1$ in (2.2.1), getting

$$(\lambda_r - \lambda_s) \sum_{p=0}^{m-1} a_p y_p(\lambda_r) y_p(\lambda_s) = c_{m-1} \begin{vmatrix} y_m(\lambda_r) & y_m(\lambda_s) \\ y_{m-1}(\lambda_r) & y_{m-1}(\lambda_s) \end{vmatrix}.$$

The determinant on the right vanishes

Expanding the determinant, we have

$$y_m(\lambda_r)y_{m-1}(\lambda_s) - y_m(\lambda_s)y_{m-1}(\lambda_r).$$

From the boundary condition, we have

$$y_m(\lambda) + hy_{m-1}(\lambda) = 0.$$

We can express $y_m(\lambda)$ in terms of $y_{m-1}(\lambda)$ as

$$y_m(\lambda) = -hy_{m-1}(\lambda).$$

Substituting this into the determinant, we replace $y_m(\lambda_r)$ and $y_m(\lambda_s)$ as follows

$$y_m(\lambda_r) = -hy_{m-1}(\lambda_r), \quad y_m(\lambda_s) = -hy_{m-1}(\lambda_s).$$

Substituting these expressions back into the determinant expansion

$$\Big(- hy_{m-1}(\lambda_r)\Big)y_{m-1}(\lambda_s) - \Big(- hy_{m-1}(\lambda_s)\Big)y_{m-1}(\lambda_r).$$

Simplifying the terms, we have

$$-hy_{m-1}(\lambda_r)y_{m-1}(\lambda_s) + hy_{m-1}(\lambda_s)y_{m-1}(\lambda_r).$$

The two terms cancel each other out because they are equal in magnitude but opposite in sign.

For the case $r = s$ we have $\lambda_r = \lambda_s$ The expression for ρ_r is given by

$$\rho_r = c_{m-1}y_{m-1}(\lambda_r)\big\{y'_m(\lambda_r) + hy'_{m-1}(\lambda_r)\big\}$$

Using the given relation, we have

$$y_m(\lambda) + hy_{m-1}(\lambda) = 0.$$

We know that

$$y_m(\lambda_r) = -hy_{m-1}(\lambda_r).$$

Differentiating this expression with respect to λ, we find

$$y'_m(\lambda) + hy'_{m-1}(\lambda) = 0$$

which implies

$$y'_m(\lambda_r) = -hy'_{m-1}(\lambda_r).$$

Substituting this result into the expression for ρ_r we have

$$\rho_r = c_{m-1}y_{m-1}(\lambda_r)\big\{ - hy'_{m-1}(\lambda_r) + hy'_{m-1}(\lambda_r)\big\}$$

Simplifying the terms inside the brackets, we obtain

$$\rho_r = c_{m-1} y_{m-1}(\lambda_r) \cdot 0$$

Therefore $\rho_r = 0$. Since the sequences (2.4.1) constitute m orthogonal and nontrivial m-vectors, there will be an eigenfunction expansion. In this case if $u_0, \ldots, u_{m-1}$ is any sequence, and we define

$$v(\lambda) = \sum_{n=0}^{m-1} a_p u_p y_p(\lambda), \tag{2.4.5}$$

then

$$u_p = \sum_{r=0}^{m-1} v(\lambda_r) y_p(\lambda_r) \rho_r^{-1}, \quad p = 0, \ldots, m-1. \tag{2.4.6}$$

In addition, we have the Parseval equality

$$\sum_{r=0}^{m-1} |v(\lambda_r)|^2 \rho_r^{-1} = \sum_{p=0}^{m-1} a_p |u_p|^2. \tag{2.4.7}$$

This expansion theorem can serve as a foundation of the expansion theorem for differential equations of the second order. Now, we can conclude the proof. ■

3 Nesting Circle Analysis

Let's consider the limiting behavior of the characteristic function $f_{m,h}(\lambda)$ as $m \to \infty$,

$$f_{m,h}(\lambda) = -\frac{z_m(\lambda) + hz_{m-1}(\lambda)}{y_m(\lambda) + hy_{m-1}(\lambda)}.$$

Let $C(m, \lambda)$ to be the locus of $f_{m,h}(\lambda)$ as h describes the real axis, by taking λ to be fixed and in the upper half-plane. We characterize by $D(m, \lambda)$ the region described by $f_{m,h}(\lambda)$ when h takes all values in the upper half-plane. For example, since $y_{-1} = 0$, $y_0 = 1/c_{-1}$, $z_{-1} = 1$, $z_0 = 0$.

The characteristic function is given by

$$f_{m,h}(\lambda) = -\frac{z_m(\lambda) + hz_{m-1}(\lambda)}{y_m(\lambda) + hy_{m-1}(\lambda)}.$$

For $m = 0$, this becomes

$$f_{0,h}(\lambda) = -\frac{z_0 + hz_{-1}}{y_0 + hy_{-1}}.$$

Substituting the given values $z_0 = 0$, $z_{-1} = 1$, $y_0 = \frac{1}{c_{-1}}$, and $y_{-1} = 0$, we obtain

$$f_{0,h}(\lambda) = -\frac{0 + h \cdot 1}{\frac{1}{c_{-1}} + h \cdot 0}.$$

Thus, the result is

$$f_{0,h}(\lambda) = -c_{-1}h, \tag{3.1}$$

so that $C(0, \lambda)$ is the real axis and $D(0, \lambda)$ is the lower half-plane. Since further $y_1(\lambda) = (a_0\lambda + b_0)/(c_0c_{-1})$, and $z_1 = -c_{-1}/c_0$, then For $m = 1$, the characteristic function becomes:

$$f_{1,h}(\lambda) = -\frac{z_1 + hz_0}{y_1 + hy_0}.$$

Substituting the given values,

$$y_1(\lambda) = \frac{a_0\lambda + b_0}{c_0c_{-1}}, \quad y_0 = \frac{1}{c_{-1}}, \quad z_1 = -\frac{c_{-1}}{c_0}, \quad z_0 = 0,$$

we have

$$f_{1,h}(\lambda) = -\frac{-\frac{c_{-1}}{c_0} + h \cdot 0}{\frac{a_0\lambda + b_0}{c_0 c_{-1}} + h \cdot \frac{1}{c_{-1}}}.$$

Simplifying the terms, we obtain

$$f_{1,h}(\lambda) = -\frac{-\frac{c_{-1}}{c_0}}{\frac{a_0\lambda + b_0}{c_0 c_{-1}} + \frac{h}{c_{-1}}}.$$

Combining terms in the denominator, we get the following

$$f_{1,h}(\lambda) = -\frac{-\frac{c_{-1}}{c_0}}{\frac{a_0\lambda + b_0 + hc_0}{c_0 c_{-1}}}.$$

Simplifying further, we have corresponding characteristic function

$$f_{1,h}(\lambda) = -\left(-\frac{c_{-1}}{c_0}\right) \cdot \frac{c_0 c_{-1}}{a_0\lambda + b_0 + hc_0}.$$

The final result is

$$f_{1,h}(\lambda) = \frac{(c_{-1})^2}{a_0\lambda + b_0 + hc_0}. \tag{3.2}$$

Before moving forward, we must revisit the linear transformation of a circle.

In a linear transformation, a circle transforms into a circle, and inverse points transform into inverse points. In the particular case in which the circle becomes a straight line, inverse points become points symmetrical about the line. For let

$$\left|\frac{z-p}{z-q}\right| = k$$

be a circle (or straight line), with p and q as inverse points. Let

$$w = \frac{az+b}{cz+d}, \quad z = \frac{dw-b}{-cw+a}.$$

Then the circle transforms into

$$\left|\frac{dw-b-p(-cw+a)}{dw-b-q(-cw+a)}\right| = k$$

or

$$\left| \frac{w - \frac{ap+b}{cp+d}}{w - \frac{aq+b}{cq+d}} \right| = k \left| \frac{cq+d}{cp+d} \right| .$$

Going back to our main discussion, for fixed λ in the upper half-plane and varying real h, $f_{1,h}(\lambda)$ describes a finite curve which must be a circle, by the elementary theory of conformal mapping that we revisited. Thus $C(1, \lambda)$ is a circle; since $\operatorname{Im} \lambda > 0$. From (3.2) we know that the characteristic function is given as

$$f_{1,h}(\lambda) = \frac{(c_{-1})^2}{a_0\lambda + b_0 + c_0 h}.$$

Here, $\lambda = u + iv$, where u is the real part, and $v > 0$ is the imaginary part of λ ($\operatorname{Im} \lambda = v > 0$). Since the numerator $(c_{-1})^2$ is constant, the properties of $f_{1,h}(\lambda)$ are determined by the denominator $a_0\lambda + b_0 + c_0 h$. The denominator can be written as

$$a_0\lambda + b_0 + c_0 h = a_0(u + iv) + b_0 + c_0 h = a_0 u + b_0 + c_0 h + i a_0 v$$

Here, the real part of the denominator is

$$\operatorname{Re}(a_0\lambda + b_0 + c_0 h) = a_0 u + b_0 + c_0 h$$

and the imaginary part is

$$\operatorname{Im}(a_0\lambda + b_0 + c_0 h) = a_0 v.$$

The imaginary part of $f_{1,h}(\lambda)$ is found using the general rule

$$\operatorname{Im}\left(\frac{1}{z}\right) = -\frac{\operatorname{Im}(z)}{|z|^2}$$

where $z = a_0\lambda + b_0 + c_0 h$. Substituting this, we get

$$\operatorname{Im} f_{1,h}(\lambda) = -\frac{\operatorname{Im}(a_0\lambda + b_0 + c_0 h)}{|a_0\lambda + b_0 + c_0 h|^2}$$

Since $\operatorname{Im}(a_0\lambda + b_0 + c_0 h) = a_0 v$ and $v > 0$, it follows that $\operatorname{Im}(a_0\lambda + b_0 + c_0 h) > 0$. Therefore, the negative sign in the expression ensures

$$\operatorname{Im} f_{1,h}(\lambda) = -\frac{\operatorname{Im}(a_0\lambda + b_0 + c_0 h)}{|a_0\lambda + b_0 + c_0 h|^2} \leq 0$$

Thus, when $\operatorname{Im} \lambda > 0$, the imaginary part of $f_{1,h}(\lambda)$ is always less than or equal to zero, which means that $f_{1,h}(\lambda)$ lies in the lower half-plane.

Equality holds only when $h \to \infty$, as this makes the imaginary part of the denominator approach zero.

Thus the circle $C(1, \lambda)$ lies in the lower half-plane, touching the real axis at the origin.

Since, again by (3.2), $f_{1,h}(\lambda)$ is finite when $\operatorname{Im} h > 0$, the region $D(1, \lambda)$ is the inside of the circle $C(1, \lambda)$. We have here the beginnings of the nesting property, in that $C(1, \lambda)$ lies inside the region $D(0, \lambda)$, and contains $D(1, \lambda)$ as its interior.

For the general result we proceed inductively, showing that

$$C(m+1, \lambda) \subset D(m, \lambda).$$

By the recurrence relation we have

$$\frac{z_{m+1}(\lambda) + hz_m(\lambda)}{y_{m+1}(\lambda) + hy_m(\lambda)} = \frac{(a_m\lambda + b_m + hc_m)z_m(\lambda) - c_{m-1}z_{m-1}(\lambda)}{(a_m\lambda + b_m + hc_m)y_m(\lambda) - c_{m-1}y_{m-1}(\lambda)}, \tag{3.3}$$

We start by analyzing the numerator and denominator of the right-hand side separately. For the numerator

$$(a_m\lambda + b_m + hc_m)z_m(\lambda) - c_{m-1}z_{m-1}(\lambda).$$

Substituting $h' = -\frac{c_{m-1}}{a_m\lambda+b_m+hc_m}$, we replace c_{m-1} with $-h'(a_m\lambda + b_m + hc_m)$. This results in

$$(a_m\lambda+b_m+hc_m)z_m(\lambda)-c_{m-1}z_{m-1}(\lambda) = (a_m\lambda+b_m+hc_m)\left[z_m(\lambda) - h'z_{m-1}(\lambda)\right].$$

For the denominator, we get

$$(a_m\lambda + b_m + hc_m)y_m(\lambda) - c_{m-1}y_{m-1}(\lambda).$$

Similarly, substituting $c_{m-1} = -h'(a_m\lambda + b_m + hc_m)$ into the denominator, we get

$$(a_m\lambda+b_m+hc_m)y_m(\lambda)-c_{m-1}y_{m-1}(\lambda) = (a_m\lambda+b_m+hc_m)\left[y_m(\lambda) - h'y_{m-1}(\lambda)\right].$$

With both the numerator and denominator simplified, the expression becomes

$$\frac{(a_m\lambda + b_m + hc_m)\left[z_m(\lambda) - h'z_{m-1}(\lambda)\right]}{(a_m\lambda + b_m + hc_m)\left[y_m(\lambda) - h'y_{m-1}(\lambda)\right]}.$$

The common factor $(a_m\lambda + b_m + hc_m)$ cancels out, leaving identity is following

$$\frac{z_m(\lambda) - h'z_{m-1}(\lambda)}{y_m(\lambda) - h'y_{m-1}(\lambda)}.$$

This is the definition of $f_{m,h'}(\lambda)$

$$f_{m,h'}(\lambda) = \frac{z_m(\lambda) - h' z_{m-1}(\lambda)}{y_m(\lambda) - h' y_{m-1}(\lambda)}.$$

Thus, we have shown

$$f_{m+1,h}(\lambda) = f_{m,h'}(\lambda), \tag{3.4}$$

where

$$h' = -c_{m-1}/(a_m\lambda + b_m + hc_m). \tag{3.5}$$

If h is real, and $\operatorname{Im}\lambda > 0$, we shall have $\operatorname{Im} h' > 0$, and so the points of $f_{m+1,h}(\lambda)$ when h is real are points of $f_{m,h'}(\lambda)$ when h' is in the upper half-plane. This proves that $D(m+1,\lambda) \subset D(m,\lambda)$, since if $\operatorname{Im} h > 0$, then (3.5) shows that $\operatorname{Im} h' > 0$.

Thus $C(2,\lambda)$ lies inside $D(1,\lambda)$, and must in particular be a circle rather than a straight line, and $D(2,\lambda)$ lying inside $D(1,\lambda)$ must be the finite region bounded by $C(2,\lambda)$, and so a disk, and so on.

We recognize two possibilities, according to whether the nesting circles contract to a point, or to a limiting circle, these two cases being the limit-point and limit-circle cases, respectively.

For this purpose we note that one point of $C(m,\lambda)$, given by $h=\infty$, is $-z_{m-1}(\lambda)/y_{m-1}(\lambda)$, so that the radius will be half the distance from this point to the furthest point of the circle, namely

$$\frac{1}{2}\max_h \left| z_{m-1}(\lambda)/y_{m-1}(\lambda) - \{z_m(\lambda) + hz_{m-1}(\lambda)\}/\{y_m(\lambda) + hy_{m-1}(\lambda)\} \right|.$$

The second term can be written as

$$\frac{z_m(\lambda) + hz_{m-1}(\lambda)}{y_m(\lambda) + hy_{m-1}(\lambda)} = \frac{z_m(\lambda)y_{m-1}(\lambda) + hz_{m-1}(\lambda)y_{m-1}(\lambda)}{y_m(\lambda)y_{m-1}(\lambda) + hy_{m-1}^2(\lambda)}.$$

Subtracting the two terms gives

$$\frac{z_{m-1}(\lambda)}{y_{m-1}(\lambda)} - \frac{z_m(\lambda) + hz_{m-1}(\lambda)}{y_m(\lambda) + hy_{m-1}(\lambda)} =$$

$$\frac{z_{m-1}(\lambda)(y_m(\lambda) + hy_{m-1}(\lambda)) - (z_m(\lambda) + hz_{m-1}(\lambda))y_{m-1}(\lambda)}{y_{m-1}(\lambda)(y_m(\lambda) + hy_{m-1}(\lambda))}.$$

Expanding the numerator yields that

$$z_{m-1}(\lambda)y_m(\lambda) + hz_{m-1}(\lambda)y_{m-1}(\lambda) - z_m(\lambda)y_{m-1}(\lambda) - hz_{m-1}(\lambda)y_{m-1}(\lambda).$$

The h-dependent terms cancel, leaving that

$$\frac{z_{m-1}(\lambda)y_m(\lambda) - z_m(\lambda)y_{m-1}(\lambda)}{y_{m-1}(\lambda)(y_m(\lambda) + hy_{m-1}(\lambda))}.$$

By using the identity below

$$c_{m-1}\{y_m(\lambda)z_{m-1}(\lambda) - z_m(\lambda)y_{m-1}(\lambda)\} = 1$$

We obtain that

$$z_{m-1}(\lambda)y_m(\lambda) - z_m(\lambda)y_{m-1}(\lambda) = \frac{1}{c_{m-1}}.$$

Substituting this into the expression gives the following

$$\frac{\frac{1}{c_{m-1}}}{y_{m-1}(\lambda)(y_m(\lambda) + hy_{m-1}(\lambda))}.$$

Simplifying, the final expression becomes

$$\frac{1}{2}\max_h |c_{m-1}y_{m-1}(\lambda)\{y_m(\lambda) + hy_{m-1}(\lambda)\}|^{-1}.$$

The maximum is reached when $|y_m(\lambda) + hy_{m-1}(\lambda)|$ has a minimum, for real h, and straightforward calculations show that this occurs when

$$h = -\mathrm{Re}\{y_m(\lambda)y_{m-1}(\lambda)\}/|y_{m-1}(\lambda)|^2,$$

The radius being then $|c_{m-1}y_m(\lambda)y_{m-1}(\lambda) - y_m(\lambda)y_{m-1}(\lambda)|^{-1}$.

4 Conclusion

This project explored the mathematical structure of discrete boundary problems and their parallels to classical Sturm-Liouville theory. By analyzing recurrence relations, oscillatory behavior, orthogonality conditions, and spectral characteristics, we developed a deeper understanding of these systems and their unique features.

The study highlighted how recurrence relations govern the dynamics of solutions, including the distribution of zeros and eigenvalues. The orthogonality of eigenfunctions played a key role in linking discrete and continuous methods, showcasing the unity of mathematical principles across domains. Additionally, spectral analysis revealed critical insights into stability and resonance phenomena.

The geometric perspective, particularly through conformal mapping and nesting circle analysis, provided a visual understanding of the behavior of characteristic functions. This approach uncovered structural richness and deepened our appreciation for the interplay between algebraic and geometric properties.

5 References

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