CONFORMABLE FRACTIONAL DERIVATIVES

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Abstract

Conformable fractional derivatives offer a novel approach to fractional calculus, departing from traditional definitions like Riemann-Liouville and Caputo. By employing the standard limit definition of the derivative, they exhibit a simpler structure and retain several properties of classical calculus, such as the chain rule and product rule. This simplicity enhances their applicability in various fields, including modeling viscoelastic materials, describing anomalous diffusion, and solving fractional differential equations. However, limitations such as the lack of semi-group property and ongoing debates regarding their physical interpretation warrant further investigation. This project provides an overview of conformable fractional derivatives, highlighting their key features.

1 Introduction

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking derivatives and integrals of functions to any arbitrary order, not just integer orders. It extends the classical definition of differentiation and integration, which are limited to integer orders, to include fractional orders.

The concept of fractional calculus dates back to the late 17th century, when mathematicians like Gottfried Wilhelm Leibniz and Niels Henrik Abel began to explore the idea of taking derivatives and integrals to non-integer orders. However, it was not until the 20th century that fractional calculus began to be more widely studied and applied.

One of the main challenges in fractional calculus is defining what it means to take a derivative or integral to a non-integer order. There are several different definitions of fractional derivatives and integrals, each with its own advantages and disadvantages. Some of the most common definitions include the Riemann-Liouville definition, the Caputo definition, and the Grünwald-Letnikov definition.

Despite the challenges, fractional calculus has a wide range of applications in various fields, including physics, engineering, and finance. For example, fractional calculus can be used to model systems with memory or hereditary properties, such as viscoelastic materials and financial markets. It can also be used to solve differential equations that cannot be solved using classical methods.

In recent years, there has been a resurgence of interest in fractional calculus, due in part to its potential applications in areas such as control theory, signal processing, and image processing. As our understanding of fractional calculus grows, it is likely that we will find even more applications for this powerful mathematical tool.

The traditional definitions of the fractional derivatives are nonlocal. This nonlocality comes from the kernel the definitions contain.

For the sake of simplifying the applicability of the fractional operators, researchers proposed local fractional operators such as the fractal and conformable derivatives. The key difference between nonlocal and local fractional operators lies in their dependence on the function's behavior over its entire domain versus just a small neighborhood of a point. Local fractional operators, like the standard derivative, only consider the function's values in an infinitesimally small region around a specific point. This makes them suitable for describing phenomena where local behavior dominates, such as classical mechanics on smooth surfaces. In contrast, nonlocal fractional operators, such as the Riemann-Liouville or Caputo fractional derivatives, take into account the function's values over its entire domain or a significant portion of it. This nonlocal characteristic makes them ideal for modeling systems with memory or long-range interactions, like viscoelastic materials or anomalous diffusion.

The choice between local and nonlocal fractional operators depends on the specific problem and the nature of the system being modeled. If the system exhibits strong local behavior, a local fractional operator might be more appropriate. However, if nonlocal effects are significant, a nonlocal fractional operator is often necessary to accurately capture the system's dynamics.

In this work, we focus on the study of one type of the local fractional derivatives which is known as the conformable fractional derivative.

Conformable fractional derivatives represent a relatively new approach to fractional calculus. They were introduced in 2014 and aim to provide a simpler and more intuitive definition compared to traditional fractional derivatives like Riemann-Liouville or Caputo.

The key feature of conformable fractional derivatives is their reliance on the standard limit definition of the derivative. This means they retain many of the familiar properties of the classical derivative, such as the chain rule and product rule, which can be challenging to establish for other fractional derivatives. This simplicity has made conformable fractional derivatives attractive for researchers and engineers seeking to apply fractional calculus to real-world problems.

However, conformable fractional derivatives also have some limitations. For instance, they do not always satisfy the semigroup property, which is a fundamental property of fractional operators. Additionally, their connection to traditional fractional calculus and their physical interpretation are still areas of ongoing research.

Despite these limitations, conformable fractional derivatives have shown promise in various applications, including modeling viscoelastic materials, describing anomalous diffusion, and solving certain types of fractional differential equations. As research progresses, a deeper understanding of their properties and limitations will continue to shape their role in the field of fractional calculus.

2 What is a conformable derivative

Let $f : [0, \infty) \to \mathbb{R}$ and $t > 0$. Then the definition its derivative of $\dfrac{df}{dt} = \lim\limits_{\varepsilon \to 0} \dfrac{f(t+\varepsilon) - f(t)}{\varepsilon}$. Accordingly, $\frac{dt^n}{dt} = nt^{n-1}$. So the question is: Can one put a similar definition for the fractional derivative of order α, where $0 < \alpha \leq 1$? Or in general for $\alpha \in (n, n+1]$ where $n \in \mathbb{N}$.

Let T_α denote the operator which is called the fractional derivative of order α. For $\alpha = 1, T_1$ satisfies the following properties:
(i) $T_1(af + bg) = aT_1(g) + bT_1(f)$, for all $a, b \in \mathbb{R}$ and f, g in the domain of T_1.
(ii) $T_1(t^p) = pt^{p-1}$ for all $p \in \mathbb{R}$.
(iii) $T_1(fg) = fT_1(g) + gT_1(f)$.
(iv) $T_1\left(\frac{f}{g}\right) = \frac{gT_1(f) - fT_1(g)}{g^2}$.
(v) $T_1(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Below we present the definition of the (local) conformable fractional derivative, which is the simpler than the traditional fractional derivative of order $\alpha \in (0, 1]$. It should be remarked that the definition can be generalized to include any α.

Definition 2.1 *Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \tag{2.1}$$

for all $t > 0, \alpha \in (0, 1)$. If f is α-differentiable in some $(0, a), a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t). \tag{2.2}$$

One should notice that a function could be α-differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $T_{\frac{1}{2}}(f)(0) = \lim_{t \to 0^+} T_{\frac{1}{2}}(f)(t) = 1$, where $T_{\frac{1}{2}}(f)(t) = 1$, for $t > 0$. But $T_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

Although the most important case for the range of α is $(0, 1)$, but, what if $\alpha \in (n, n+1]$ for some natural number n ? What would be the definition?

We will, sometimes, write $f^{(\alpha)}(t)$ for $T_\alpha(f)(t)$, to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

The connection between α-differentiability and continuity is presented in the following theorem.

Theorem 2.1 *If a function $f : [0, \infty) \longrightarrow \mathbb{R}$ is α-differentiable at $t_0 > 0, \alpha \in (0, 1]$, then f is continuous at t_0.*

Proof. Since $f\left(t_0 + \varepsilon t_0^{1-\alpha}\right) - f(t_0) = \frac{f\left(t_0+\varepsilon t_0^{1-\alpha}\right)-f(t_0)}{\varepsilon}\varepsilon$. *Then,*

$$\lim_{\varepsilon\to 0}\left[f\left(t_0 + \varepsilon t_0^{1-\alpha}\right) - f(t_0)\right] = \lim_{\varepsilon\to 0}\frac{f\left(t_0 + \varepsilon t_0^{1-\alpha}\right) - f(t_0)}{\varepsilon} \cdot \lim_{\varepsilon\to 0}\varepsilon$$

Let $h = \varepsilon t_0^{1-\alpha}$. *Then,*

$$\lim_{h\to 0}\left[f(t_0 + h) - f(t_0)\right] = f^{(\alpha)}(t_0) \cdot 0$$

which implies that

$$\lim_{h\to 0} f(t_0 + h) = f(t_0)$$

Hence, f is continuous at t_0.

It can easily observed that the conformable derivative satisfies the properties of the classical derivative as mention in the following theorem.

Theorem 2.2 *Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $t > 0$. Then*

1. $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.
2. $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.
3. $T_\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$.
4. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
5. $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
6. *If, in addition, f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Proof 2.1

$$\begin{aligned} T_\alpha(af+bg) &= \lim_{\varepsilon\to 0} \tfrac{(af+bg)(t+\varepsilon t^{1-\alpha}-(af+bg)(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \tfrac{af(t+\varepsilon t^{1-\alpha}+bg(t+\varepsilon t^{1-\alpha}-af(t)+bg(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \tfrac{a[f(t+\varepsilon t^{1-\alpha})-f(t)]}{\varepsilon} + \lim_{\varepsilon\to 0} \tfrac{b[g(t+\varepsilon t^{1-\alpha})-g(t)]}{\varepsilon} \\ &= aT_\alpha(f)+bT_\alpha(g) \end{aligned}$$

$$\begin{aligned} T_\alpha(t^p)(t) &= \lim_{\varepsilon\to 0} \tfrac{(t^p)(t+\varepsilon t^{1-\alpha}-(t^p)(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \tfrac{(t+\varepsilon t^{1-\alpha})^p-t^p}{\varepsilon} \end{aligned}$$

let $h=\varepsilon t^{1-\alpha}$, *then* $\varepsilon = t^{1-p}h$ *therefore*

$$\begin{aligned} T_\alpha(t^p)(t) &= \lim_{h\to 0} \tfrac{(t+h)^p-t^p}{ht^{\alpha-p}} \\ &= t^{1-\alpha}\lim_{h\to 0} \tfrac{(t+h)^p-t^p}{h} \\ &= t^{1-\alpha}pt^{p-1} \\ &= pt^{p-\alpha} \end{aligned}$$

$f(t)=\lambda$ *then*

$$\begin{aligned} T_\alpha(\lambda)(t) &= \lim_{\varepsilon\to 0} \tfrac{\lambda(t+\varepsilon t^{1-\alpha})-\lambda(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \tfrac{\lambda-\lambda}{\varepsilon} = \lim_{\varepsilon\to 0} 0 = 0 \end{aligned}$$

$$
\begin{aligned}
T_\alpha(fg)(t) &= \lim_{\varepsilon\to 0} \tfrac{(fg+\varepsilon t^{1-\alpha})-(fg)(t)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \tfrac{f(t+\varepsilon t^{1-\alpha})g(t+\varepsilon t^{1-\alpha})-f(t)g(t)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \tfrac{f(t+\varepsilon t^{1-\alpha})g(t+\varepsilon t^{1-\alpha})-f(t)g(t+\varepsilon t^{1-\alpha})+f(t)g(t+\varepsilon t^{1-\alpha})-f(t)g(t)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \tfrac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} g(t+\varepsilon t^{1-\alpha}) + f(t)\lim_{\varepsilon\to 0} \tfrac{g(t+\varepsilon t^{1-\alpha})-g(t)}{\varepsilon} \\
&= T_\alpha(f)(t)g(t+\varepsilon t^{1-\alpha}) + f(t)T_\alpha(g)(t)
\end{aligned}
$$

Since g is continuous at t, $\lim_{\varepsilon\to 0} g(t+\varepsilon t^{1-\alpha}) = g(t)$.

Hence $T_\alpha(fg)(t) = T_\alpha(f)(t)g(t) + f(t)T_\alpha(g)(t)$.

$$
\begin{aligned}
T_\alpha(f/g)(t) &= \lim_{\varepsilon\to 0} \tfrac{(f/g)(t+\varepsilon t^{1-\alpha})-(f/g)(t)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \tfrac{\frac{(f)(t+\varepsilon t^{1-\alpha}}{g(t+\varepsilon t^{1-\alpha})})-\frac{(f)(t)}{g(t)}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \tfrac{g(t)f(t+\varepsilon t^{1-\alpha})-f(t)g(t+\varepsilon t^{1-\alpha})}{\varepsilon g(t+\varepsilon t^{1-\alpha})g(t)} \\
&= \lim_{\varepsilon\to 0} \tfrac{g(t)f(t+\varepsilon t^{1-\alpha})}{\varepsilon g(t+\varepsilon t^{1-\alpha})g(t)} - \lim_{\varepsilon\to 0} \tfrac{f(t)g(t+\varepsilon t^{1-\alpha})}{\varepsilon g(t+\varepsilon t^{1-\alpha})g(t)} \\
&= \lim_{\varepsilon\to 0} \tfrac{g(t)f(t+\varepsilon t^{1-\alpha})-f(t)g(t)}{\varepsilon g(t+\varepsilon t^{1-\alpha})g(t)} - \lim_{\varepsilon\to 0} \tfrac{f(t)g(t+\varepsilon t^{1-\alpha})-f(t)g(t)}{\varepsilon g(t+\varepsilon t^{1-\alpha})g(t)}
\end{aligned}
$$

Let f be differentiable and let $h = \varepsilon t^{1-\alpha}$.

Then $\varepsilon = t^{\alpha-1}h$. Therefore

$$
\begin{aligned}
T_\alpha(f)(t) &= \lim_{\varepsilon\to 0} \tfrac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \tfrac{f(t+h)-f(t)}{h t^{\alpha-1}} \\
&= t^{\alpha-1} \lim_{\varepsilon\to 0} \tfrac{f(t+h)-f(t)}{h} \\
&= t^{1-\alpha} \tfrac{df}{dt}(t)
\end{aligned}
$$

Definition 2.2 *Let $\alpha \in (n, n+1]$, and f be an n-differentiable at t, where $t > 0$. Then the conformable fractional derivative of f of order α is defined as*

$$
T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f^{(\lceil\alpha\rceil-1)}\left(t + \varepsilon t^{(\lceil\alpha\rceil-\alpha)}\right) - f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon}
$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Remark 2.1. *As a consequence of Definition 2.2, one can easily show that*

$$
T_\alpha(f)(t) = t^{(\lceil\alpha\rceil-\alpha)} f^{\lceil\alpha\rceil}(t)
$$

where $\alpha \in (n, n+1]$, and f is $(n+1)$-differentiable at $t > 0$.

3 Conformable fractional derivatives of certain functions

Theorem 3.1 *Let $\alpha \in (0,1]$. The following assertions hold.*

1. $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.
2. $T_\alpha(1) = 0$.
3. $T_\alpha(e^{cx}) = cx^{1-\alpha}e^{cx}, c \in \mathbb{R}$.
4. $T_\alpha(\sin bx) = bx^{1-\alpha}\cos bx, b \in \mathbb{R}$.
5. $T_\alpha(\cos bx) = -bx^{1-\alpha}\sin bx, b \in \mathbb{R}$.
6. $T_\alpha\left(\frac{1}{\alpha}t^\alpha\right) = 1$.
7. $T_\alpha\left(\sin\frac{1}{\alpha}t^\alpha\right) = \cos\frac{1}{\alpha}t^\alpha$.
8. $T_\alpha\left(\cos\frac{1}{\alpha}t^\alpha\right) = -\sin\frac{1}{\alpha}t^\alpha$.
9. $T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t^\alpha}$.

Proof 3.1

$$\begin{aligned} T_\alpha(t^p) &= \lim_{\varepsilon\to 0}\tfrac{t^p(t+\varepsilon t^{1-\alpha})-(t^p)(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\tfrac{(t+\varepsilon t^{1-\alpha})-t^p}{\varepsilon} \end{aligned}$$

Let $h = \varepsilon t^{1-\alpha}$ *then* $\varepsilon = t^{1-\alpha}h$ *. Therefore*

$$\begin{aligned} T_\alpha(t^p) &= \lim_{\varepsilon\to 0}\tfrac{(t+\varepsilon t^{1-\alpha})-t^p}{\varepsilon} \\ &= \lim_{h\to 0}\tfrac{(t+h)^p-t^p}{ht^{\alpha-1}} \\ &= t^{1-\alpha}\lim_{h\to 0}\tfrac{(t+h)^p-t^p}{h} \\ &= t^{1-\alpha}(t^p)' \\ &= t^{1-\alpha}pt^{p-1} \\ &= pt^{p-\alpha} \end{aligned}$$

From Theorem 2.2 (3), for all constant functions: $f(t) = \lambda$, $T_\alpha(\lambda) = 0$ *since* $\lambda = 1, f(t) = 1$ *for constant function* $T_\alpha(1) = 0$.

$$\begin{aligned} T_\alpha(e^{cx}) &= \lim_{\varepsilon\to 0} \tfrac{c(x+\varepsilon x^{1-\alpha})-e^{cx}}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \tfrac{e^{cx}e^{\varepsilon cx}-e^{cx}}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \tfrac{e^{cx}(e^{c\varepsilon x}-1)}{\varepsilon} \end{aligned}$$

Let $c\varepsilon x^{1-\alpha} = h$, *then* $\varepsilon = \frac{x^{\alpha-1}h}{c}$

$$\begin{aligned} \lim_{h\to 0} \tfrac{e^{cx}(e^h-1)}{\frac{x^{\alpha-1}h}{c}} &= ce^{cx}x^{1-\alpha} \lim_{h\to 0} \tfrac{e^h-1}{h} \\ &= ce^{cx}x^{1-\alpha} \end{aligned}$$

$$\begin{aligned} T_\alpha(sinbx) &= \lim_{\varepsilon\to 0} \tfrac{sin(b(x+\varepsilon x^{1-\alpha}))-sinbx}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \tfrac{sin(bx+b\varepsilon x^{1-\alpha}))-sinbx}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \tfrac{sin(bx)cos(b\varepsilon x^{1-\alpha})+cos(bx)sin(b\varepsilon x^{1-\alpha})-sin(bx)}{\varepsilon} \end{aligned}$$

Let $h = b\varepsilon x^{1-\alpha}$ *then* $\varepsilon = \frac{x^{\alpha-1}h}{b}$.

$$\begin{aligned} \lim_{h\to 0} \frac{sin(bx)cosh + cos(bx)sinh - sin(bx)}{\frac{x^{\alpha-1}h}{b}} &= bx^{1-\alpha} \lim_{h\to 0} sin(bx)\frac{cosh}{h} + cosbx\frac{sinh}{h}\frac{sinbx}{h} \\ &= bx^{1-\alpha}[\lim_{h\to 0} cosh \lim_{h\to 0} sinhx \\ &+ cosx \lim_{h\to 0} \frac{sinh}{h} - \lim_{h\to 0} \frac{sinbx}{h}] \\ &= bx^{1-\alpha}cosbx \end{aligned}$$

$$
\begin{aligned}
T_\alpha(cosbx) &= \lim_{\varepsilon\to 0}\frac{cos(b(x+\varepsilon x^{1-\alpha}))-cosbx}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{cos(bx+b\varepsilon x^{1-\alpha})-cosbx}{\varepsilon}
\end{aligned}
$$

Let $b\varepsilon x^{1-\alpha}=h$ *, then* $\varepsilon=\frac{x^{\alpha-1}h}{b}$*. Therefore,*

$$
\begin{aligned}
T_\alpha(cosbx) &= \lim_{h\to 0}\frac{cos(bx+h)-cosbx}{\frac{x^{\alpha-1}h}{b}}\\
&= (bx^{1-\alpha})\lim_{h\to 0}\frac{cos(bx+h)-cosbx}{h}\\
&= bx^{1-\alpha}\lim_{h\to 0}\frac{cosbxcosh-sinbxsinh-cosbx}{h}\\
&= bx^{1-\alpha}[\lim_{h\to 0}cosh\lim_{h\to 0}\frac{cos(bx)}{h}-sinbx\lim_{h\to 0}\frac{sinh}{h}-\lim_{h\to 0}\frac{cosbx}{h}]\\
&= -bx^{1-\alpha}sinbx
\end{aligned}
$$

$T_\alpha(\frac{1}{\alpha}t^\alpha)=1$ *, from Theorem 2.2(2),* $T_\alpha(t^p)=pt^{p-\alpha}$ *for all* $p\in\mathbb{R}$ *. Let*

$p = \alpha \ then \ T_\alpha(t^\alpha) = \alpha t^{\alpha-\alpha} = \alpha$

$$\begin{aligned} T_\alpha(\frac{1}{\alpha}t^\alpha) &= \lim_{\varepsilon\to 0}\frac{\frac{1}{\alpha}(t+\varepsilon t^{1-\alpha})-\frac{1}{\alpha}t^\alpha}{\varepsilon} \\ &= \frac{1}{\alpha}\lim_{\varepsilon\to 0}\frac{(t+\varepsilon t^{1-\alpha})-t^\alpha}{\varepsilon} \\ &= \frac{1}{\alpha}T_\alpha(t^\alpha) \\ &= \frac{1}{\alpha}\alpha \\ &= 1 \end{aligned}$$

$T_\alpha\left(\sin\frac{1}{\alpha}t^\alpha\right) = \cos\frac{1}{\alpha}t^\alpha \ because \ g = \frac{1}{\alpha}t^\alpha \ , \ f = sin(t).$

$$\begin{aligned} T_\alpha(fog)(t) &= t^{1-\alpha}\frac{d(sin(t))}{dt}\frac{dg}{dt} \\ &= t^{1-\alpha}cos(\frac{1}{\alpha}t^\alpha)(\frac{1}{\alpha}t^\alpha)' \\ &= t^{1-\alpha}\frac{1}{\alpha}t^\alpha\alpha t^{\alpha-1}cos(\frac{1}{\alpha}t^\alpha) \\ &= cos(\frac{1}{\alpha}t^\alpha) \end{aligned}$$

$$\begin{aligned} rclT_\alpha\left(\cos\frac{1}{\alpha}t^\alpha\right) &= t^{1-\alpha}(-sin(\frac{1}{\alpha}t^\alpha))(\frac{1}{\alpha}t^\alpha)' \\ &= t^{1-\alpha}\alpha\frac{1}{\alpha}t^{\alpha-1}(-sin(\frac{1}{\alpha}t^\alpha)) \\ &= -sin(\frac{1}{\alpha}t^\alpha) \end{aligned}$$

$$
\begin{aligned}
T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) &= T_\alpha(fog)(t), \textit{where} \;\; f = e^t, g = \frac{1}{\alpha}t^\alpha \\
&= t^{1-\alpha}e^{\frac{1}{\alpha}t^\alpha}(\frac{1}{\alpha}t^\alpha)' \\
&= t^{1-\alpha}\frac{1}{\alpha}\alpha t^{\alpha-1}e^{\frac{1}{\alpha}t^\alpha}
\end{aligned}
$$

4 Mean value theorem in the conformable derivative context

It is possible to prove basic analysis theorems like the Rolle's theorem and the mean value theorem in the framework of the conformable derivative.

Theorem 4.1 *(Rolle's Theorem for Conformable Fractional Differentiable Functions). Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies*

(i) f is continuous on $[a, b]$,

(ii) f is α-differentiable for some $\alpha \in (0, 1)$,

(iii) $f(a) = f(b)$.

Then, there exists $c \in (a, b)$, such that $f^{(\alpha)}(c) = 0$.

Proof 4.1 *Since f is continuous on $[a, b]$, and $f(a) = f(b)$, there is $c \in (a, b)$, which is a point of local extrema. With no loss of generality, assume c is a point of local minimum. So*

$$f^{(\alpha)}(c) = \lim_{\varepsilon \to 0^+} \frac{f\left(c + \varepsilon c^{1-\alpha}\right) - f(c)}{\varepsilon} = \lim_{\varepsilon \to 0^-} \frac{f\left(c + \varepsilon c^{1-\alpha}\right) - f(c)}{\varepsilon}$$

But, the first limit is non-negative, and the second limit is non-positive. Hence $f^{(\alpha)}(c) = 0$.

Theorem 4.2 *(Mean Value Theorem for Conformable Fractional Differentiable Functions). Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies*
(i) f is continuous on $[a, b]$.
(ii) f is α-differentiable for some $\alpha \in (0, 1)$.

Then, there exists $c \in (a, b)$, such that $f^{(\alpha)}(c) = \frac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}$.

Proof 4.2 *Consider the function*

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{\frac{1}{\alpha}b^{\alpha} - \frac{1}{\alpha}a^{\alpha}} \left(\frac{1}{\alpha}x^{\alpha} - \frac{1}{\alpha}a^{\alpha} \right)$$

Then the function g satisfies the conditions of Rolle's theorem. Hence there exists $c \in (a, b)$, such that $g^{(\alpha)}(c) = 0$. Using the fact that $T_{\alpha}\left(\frac{1}{\alpha}t^{\alpha}\right) = 1$, the result follows.

5 The conformable fractional integral

When it comes to integration, the most important class of functions to define the integral is the space of continuous functions.

So, using the Weierstrass theorem, it is enough to define the fractional integral on polynomials. This suggests the following.

Let $\alpha \in (0, \infty)$. Define $I_\alpha\left(t^p\right) = \frac{t^{p+\alpha}}{p+\alpha}$ for any $p \in \mathbb{R}$, and $\alpha \neq -p$.
If $f(t) = \sum_{k=0}^{n} b_k t^k$, then we define $I_\alpha(f) = \sum_{k=0}^{n} b_k J_\alpha\left(t^k\right) = \sum_{k=0}^{n} b_k \frac{t^{k+\alpha}}{k+\alpha}$.
If $f(t) = \sum_{k=0}^{\infty} b_k t^k$, where the series is uniformly convergent, then we define $I_\alpha(f) = \sum_{k=0}^{\infty} b_k \frac{t^{k+\alpha}}{k+\alpha}$.
Clearly, I_α is linear on its domain. Further, if $\alpha = 1$, then J_α is the usual integral.

Example 5.1 *if $\alpha = \frac{1}{2}$, then*

$$I_\alpha(\sin t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+\frac{3}{2}}}{\left(2n+\frac{3}{2}\right)(2n+1)!}$$

Similarly one can find the fractional integral of $\cos t$ and e^t, and for any $\alpha \in (0, 1)$.
These examples suggest the following definition for the α-fractional integral of a function f starting from $\mathbf{a} \geq 0$.

Definition 5.1 *$I_\alpha^a(f)(t) = I_1^a\left(t^{\alpha-1} f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$, where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$.*
So, $I_{\frac{1}{2}}^0(\sqrt{t}\cos t) = \int_0^t \cos x dx = \sin t$, and $I_{\frac{1}{2}}^0(\cos 2\sqrt{t}) = \sin 2\sqrt{t}$.
One of the nice results is the following.

Theorem 5.1 *$T_\alpha I_\alpha^a(f)(t) = f(t)$, for $t \geq a$, where f is any continuous function in the domain of I_α.*

Proof 5.1 *Since f is continuous, then $I_\alpha^a(f)(t)$ is clearly differentiable. Hence,*

$$
\begin{aligned}
T_\alpha\left(I_\alpha^a(f)\right)(t) &= t^{1-\alpha}\frac{d}{dt}I_\alpha^a(f)(t)\\
&= t^{1-\alpha}\frac{d}{dt}\int_a^t \frac{f(x)}{x^{1-\alpha}}dx\\
&= t^{1-\alpha}\frac{f(t)}{t^{1-\alpha}}\\
&= f(t).
\end{aligned}
$$

6 Differential equations in the conformable fractional settings

In this section, methods to solve differential equations involving conformable derivatives are discussed

Let us consider the following equation

$$T_\alpha y + h(x)y = k(x), \tag{6.1}$$

where $0 < \alpha < 1$ and $h, k : R \to R$ are α-differentiable functions. If $k(t) = 0$, then (6.1) is said to be homogeneous. Otherwise, it is called non-homogeneous.

Theorem 6.1 *The solution of the homogeneous conformable differential equation corresponding to (6.1) reads*

$$y_h(x) = ce^{-I^0_\alpha h(x)}. \tag{6.2}$$

Proof 6.1 *It is enough to show that the homogeneous equation associated with (6.1) is fulfilled by the function.* $y_h(x) = ce^{-I^0_\alpha h(x)}$*. Now, replacing* y_h *in the equation we have*

$$\begin{aligned} T_\alpha y + h(x)y &= ct^{1-\alpha}\frac{d}{dx}\left[e^{-I^0_\alpha h(x)}\right] + \mathrm{ch}(x)e^{-I^0_\alpha h(x)} \\ &= -cx^{1-\alpha}\frac{d}{dx}\left[I^0_\alpha h(x)\right]e^{-I^0_\alpha h(x)} + \mathrm{ch}(x)e^{-I^0_\alpha h(x)} \\ &= -cx^{1-\alpha}\frac{h(x)}{x^{1-\alpha}}e^{-I^0_\alpha h(x)} + \mathrm{ch}(x)e^{-I^0_\alpha h(x)} \\ &= 0. \end{aligned}$$

Theorem 6.2 *The particular solution of the conformable differential equation (6.1) can be found by using the method of variation of parameters and it is given by*

$$y_p(x) = \lambda(x)e^{-I^0_\alpha h(x)}, \tag{6.3}$$

where h *is obtained through the following condition*

$$\lambda(x) = I^0_\alpha\left(k(x)e^{I^0_\alpha h(x)}\right). \tag{6.4}$$

Proof 6.2 *Plugging y_p in (6.1) and using the fourth property in Theorem 2.2 we get*

$$T_\alpha(\lambda(x))e^{-I_\alpha^0 h(x)} + \lambda(x)T_\alpha(e^{-I_\alpha^0 h(x)}) + h(x)\lambda(x)e^{-I_\alpha^0 h(x)} = k(x).$$

Since $e^{-I_\alpha^0 h(x)}$ is a solution of the homogeneous equation, we have $T_\alpha(e^{-I_\alpha^0 h(x)}) = -h(x)e^{-I_\alpha^0 h(x)}$. Therefore, we have

$$T_\alpha(\lambda(x))e^{-I_\alpha^0 h(x)} = k(x).$$

That is,

$$T_\alpha(\lambda(x)) = k(x)e^{I_\alpha^0 h(x)}.$$

The result is then obtained by performing the conformable fractional integral on both sides of the previous example.

Remark 6.1 *The general solution to the differential equations defined by (6.1) is given as $y(x) = y_h(x) + y_p(x)$.*

Example 6.1 *Lets find the general solution*

$$T_{\frac{1}{2}}y + \sqrt{x}y = xe^{-x}.$$

Here $\alpha = \frac{1}{2}, h(x) = \sqrt{x}$ and $k(x) = xe^{-x}$. Thus, $I_{\frac{1}{2}}^0 h(x) = \int_0^x t^{\frac{1}{2}-1}\sqrt{t}dt = \int_0^x 1dt = x.$. Therefore, $y_h(x) = ce^{-x}$. Now $\lambda(x) = \int_0^x t^{\frac{1}{2}-1}te^{-t}e^t dt = \frac{2}{3}x^{\frac{3}{2}}$. Consequently, $y_p(x) = \frac{2}{3}x^{\frac{3}{2}}e^{-x}$. The general equation has the form

$$y(x) = ce^{-x} + \frac{2}{3}x^{\frac{3}{2}}e^{-x}.$$

7 Conclusion

Conformable fractional derivatives present an intriguing alternative within the realm of fractional calculus. Their foundation in the standard limit definition of the derivative lends them a degree of simplicity and intuitiveness that distinguishes them from more established approaches like Riemann-Liouville and Caputo. This simplicity translates to several advantages, including the retention of familiar properties such as the chain rule and product rule, which can significantly facilitate the analysis and solution of fractional differential equations.

However, it is crucial to acknowledge the limitations inherent to conformable fractional derivatives. The lack of adherence to the semigroup property, a fundamental characteristic of fractional operators, raises questions about their deeper connection to the broader framework of fractional calculus. Moreover, their physical interpretation and the extent to which they accurately model real-world phenomena remain subjects of ongoing research and debate.

Despite these challenges, the potential of conformable fractional derivatives is undeniable. Their applicability in diverse fields, including modeling viscoelastic materials, describing anomalous diffusion, and solving fractional differential equations, demonstrates their significance. Continued research efforts are essential to further elucidate their properties, refine their theoretical foundations, and expand their range of applications.

By addressing the existing limitations and deepening our understanding of their underlying principles, conformable fractional derivatives have the potential to become a valuable tool in the arsenal of mathematical techniques used to model and analyze complex systems in various scientific and engineering disciplines.

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