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Solve It

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Question: Let $\mathbb{K}[x_1, ..., x_n] = \{f(x_1, ..., x_n) \mid f \textit{ is a polynomial}\}$ be the polynomial ring in $\{x_1, ..., x_n\}$ over any field $\mathbb{K}$ (You can consider $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). We define for $f \in \mathbb{K}[x_1, ..., x_n]$:

$$V(f) = \{p = (a_1, ..., a_n) \mid f(p) = 0\}$$

So for any $S \subseteq \mathbb{K}[x_1, ..., x_n]$ we can define

$$V(S) = \{p = (a_1, ..., a_n) \mid f(p) = 0, \forall f \in S\}$$

.

PROVE THE FOLLOWINGS

1. $V(S) = \bigcap_{i=1}^{m} V(f_i)$ where $S = \{f_1, ..., f_m\}$.
   Hint: First prove $V(f_1, f_2) = V(f_1) \bigcap V(f_2)$ and observe the idea.

2. $\mathbb{K}^n = \{(a_1, ..., a_n) \mid a_i \in k\}$ and $\emptyset$ can be written as V of some polynomials.
   Hint: You need to find two polynomials (very easy ones) f and g such that $\mathbb{K}^n = V(f)$ and $\emptyset = V(g)$.

3. $\bigcup_{i=1}^{m} V(f_i) = V(f_1 f_2 ... f_m)$.
   Remark: Be careful that this is a finite union.

Remark: For instance you can consider a single variable polynomial ring $\mathbb{K}[x]$ over $\mathbb{K}$. Let us prove $V(f_1, f_2) = V(f_1) \bigcap V(f_1)$. Let $p \in V(f_1, f_2)$. Then $f_1(p) = 0$ and $f_2(p) = 0$ and hence $p \in V(f_1)$ and $p \in V(f_2)$ i.e, $p \in V(f_1) \bigcap V(f_2)$. We have proved that $V(f_1, f_2) \subseteq V(f_1) \bigcap V(f_2)$. Conversely, let $p \in V(f_1) \bigcap V(f_2)$. Then $p \in V(f_1)$ and $p \in V(f_2)$ and hence $f_1(p) = 0$ and $f_2(p) = 0$ i.e, $p \in V(f_1, f_2)$. This completes the proof. Using a similar idea you can easily generalize it for n variable polynomial ring $\mathbb{K}[x_1, ..., x_n]$. (The only difference appears in the form of the point $(p_1, ..., p_n)$).